SIX PHILOSOPHICAL DISSERTATIONS

ON

The Mechanical Powers.
Elastic Bodies.
Falling Bodies.
The Cycloid.
The Parabola.
The Rain-Bow.

BY

Dr. MORGAN, late Maſter of *Clare Hall.*

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ERRATA.

Page 23, l. 5, dele *the*.

23, for GB, read GD.

19, l. 16, for *positions are the same*, read, *position is the same*.

ON THE

MECHANICAL POWERS.

SINCE a body with two united forces, always deſcribes the diagonal of a Parallelogram in the ſame time as it would do the ſides, if the forces were ſeparate; it is evident, that any force whatſoever, acting in a given direction, may be looked upon as the effect of two other forces acting in directions, which at the ſame point ſhall on each ſide be any way inclined to the given direction, provided they make an angle leſs than two right ones: And this is abundantly confirmed in Mechanics, for by ſuch a reſolution of a given force into two others, the known Properties of the Mechanic Powers, ſuch as the Ballance, the inclined Plain, &c. may be eaſily deduced.

Of the BALLANCE or LEAVER. Prop. I.

If two forces, which act upon the arms of a ballance in given directions that are in the ſame plain with thoſe arms, ballance one another; theſe forces are to each other reciprocally, as perpendiculars let fall from the center of the ballance to their directions.

Plate I. Fig. 1. *Dem.* (*See Newt. Princ. pag.* 14.) Let C be the center of the Ballance, C *p*, C P the arms, E*p*, P A the directions of the forces acting upon the arms, C*p*, C P. Let C E be drawn perpendicular to *p* E, and C D to P A, meeting them in E and D. On the center C, and with the radius C E, *viz.* the longeft of the perpendiculars, let a circle be defcribed, which fhall interfect the direction of the force P in A, and let the line C A be drawn; to which let A G be drawn perpendicular, and G F parallel, meeting D P A in F.

It is evident, that the arms of the Ballance C P, C *p* may be looked upon as lines that will not bend, lying in the plain moveable about the center C; and the fame may be underftood of any other lines drawn through the center C, and lying in the fame plain. Now fince it is manifeft, that there is no difference in what points of the lines, in which the forces P and *p* act, thofe forces are placed; fince wherefoever they are in thofe lines, they will have exactly the fame power to turn the plain C D A *p* E about it's center: The forces P and *p* may be fuppofed to be in the points A and E. Then the force P, fuppofed to be in A, may be refolved (as was before obferved) into two other forces; one of which may act according to the line C A produced, and the other according to the line A G; and which may be to each other as F G to G A, but each of them fingly to P, as F G and A G fingly to A F, as will be evident if the triangle A G F be compleated in the parallelogram A G F *g*. It is alfo manifeft, that the force which is as F G, and which acts according to the line C A paffing through the center of the plain, does nothing at all towards turning that plain about the center C; but the force which is as A G, and which draws the line C A perpendicularly; fince, by the hypothefis, it ballances the force *p*, which draws the line C E, equal to C A (by conftruction) perpendicularly alfo, it muft neceffarily be equal to it. Wherefore *p* will be to P as A G to A F; or as D C (by reafon of the fimilar triangles F G A, A C D) to C A or C E; that is, the forces *p* and P are to one another reciprocally as perpendiculars let fall from the center to the lines in which they act.

Corol.

Corol. 1. If the arms lie in a ſtraight line, and the determinations of the forces be parallel, it is evident that the forces are reciprocally as the length of the arms.

2. Hence alſo, in the angular Ballance P C *p*, which turns about the immoveable center C; the ſituation which it will be in, when any two given bodies are fixed to the ends P and *p*, may be determined. For if the line P *p*, which joins the ends of the Ballance, be divided in reciprocal proportion to the weights, and the point of diviſion T be made in the line C T drawn through the center, parallel to the direction of the weights; I ſay it is done; for P D and *p* E being drawn parallel, and D C E perpendicular to C T; it is evident that D C E is divided in C, in the ſame proportion that P T *p* is in T, and that the weights may be ſuppoſed to be placed in the points D and E. Wherefore this will be the ſituation of the points P and *p*, that is, of the Ballance itſelf when the weights are in *equilibrio*. Plate I. Fig. 2.

3. In the Ballance or Leaver it is evident, that two forces, ſuch as P and *p*, which, when the Ballance librates to and fro, are reciprocally as the velocities of the points D and E, reckoned according to the directions of thoſe forces, will ballance each other. Plate I. Fig. 1.

Of the INCLINED PLAIN. Prop. II.

If a force, with a given direction, ſupports a weight upon an inclined Plain; that force is to the weight, as the ſine of the inclination of the Plain to the ſine of the angle which is made by the line in which the force acts, and the line perpendicular to the Plain.

Dem. Let A B be the inclined Plain, P the weight ſupported, D P V the direction of the force which ſupports the weight. Let P C be drawn perpendicular to A B; and from the point C let C B be drawn parallel to the horizon, and perpendicular to the common ſection of the Plain and the horizon, meeting the Plain in B; and C A perpendicular to the horizon and alſo to C B, meeting the Plain in A, and the line in which the force acts in V. Plate I. Fig. 3.

Now P may be conceived to be held unmoved by three forces acting together; one of which is the force of the weight itſelf tending downwards in a line parallel to V C; the ſecond is the force acting

acting in the line D P V; and the third is the resistance of the Plain itself, acting in the line C P perpendicular to the Plain: But these three forces are to each other (from what was said before) as the sides of the triangle V P C; as will be evident, by drawing a line through P parallel to V C, and compleating the parallelogram. The force therefore is to the weight which it sustains, as P V to V C; that is, as the sine of the angle V C P, or A B C, to the sine of the angle C P V or C P D. *Q. E. D.*

Corol. 1. If the points V and A coincide, that is, if the force acts according to the direction B A, the angle C P D will be a right angle; and therefore, in that case, the force is to the weight as the sine of the inclination of the Plain to the radius, or as the height of the Plain A C to its length A B. And in this case, the force which is required to support a given weight is least of all; because the proportion of the sine of the inclination of the Plain to the radius, is less than its proportion to any other sine whatsoever.

2. If the point V falls above A, the greater the angle A P V is, so much the more force is necessary to support the given weight upon the Plain A B. Insomuch, that by increasing the angle A P V, the proportion of the sine of the angle A B C to the sine of the angle C P D is also increased, 'till P V, A V becoming parallel, and the angles V C P, C P D for that reason equal, the force and the weight will also become equal.

3. So likewise, if the point V falls below A, as at *v*, the force requisite to support the given weight, is again increased: the angle A P *v* being increased, till P *v*, *v* C become equal, the force and the weight will become equal again. Further, when the lines P *v*, P C coincide, and the angle *v* P C by that means vanishes, the sine of the angle A B C will bear an infinite proportion to the sine of that; that is, no finite force whatsoever, acting in a line perpendicular to the plain, will be able to support the weight upon the plain.

4. If the line in which the force acts be parallel to the base of the plain, the weight is to the force which supports it as B C to C A, or as the base of the plain to the height of it.

5. If

If from the point P, P F be let fall perpendicular to B C, and from Plate I. the point C, C G perpendicular to V P, it will eaſily appear, that Fig. 4. P V is to V C (that is, the force is to the weight) as C F to C G. Wherefore the force and the weight will then ſupport one another upon an inclined plain, when they are to each other reciprocally as perpendiculars drawn from the point C to the lines in which they act; or (if G C F be looked upon as an angular ballance moveable about the center C) reciprocally as the velocities of the points G and F reckoned upon the lines in which the forces act.

Of the WEDGE. Prop. III.

If three forces acting together upon an Iſoſceles Wedge, in lines perpendicular to the three plains of the Wedge, two of which forces, *viz.* thoſe acting upon the ſides, are equal to each other, and the direction of the third, which acts upon the baſe of the Wedge, paſſes through its vertex; if, I ſay, theſe three forces ſupport each other, the force acting upon the baſe will be to the other two, as the baſe of the wedge to the ſum of its ſides.

Dem. Let A B C repreſent a wedge; and let C G be perpendicular Plate I. to A B, and G D, G *d* perpendicular to A C, B C, and theſe Fig. 5. will be the directions of the three forces. In the lines G D, G *d* produced, let D E and *d e* be taken equal to each other, which may therefore repreſent the two equal forces, which act upon the ſides in the directions E D, *e d*. Let E F, *e f* be drawn parallel to A B, and D F, *d f* parallel to G C, ſo as to form the triangles, D E F, *d e f*. Now each of the forces E D, *e d*, may be imagined to be reſolved into two other forces, which are to each other as E F to F D, and *e f* to *f d*, and to act in thoſe lines: And thoſe two, which are as E F, *e f*, becauſe they are equal and oppoſite, will deſtroy each other. But the force which acts upon the baſe A B, in the line G C; becauſe it ſupports the two other forces F D, *f d*, both which are the ſame way, and act in a contrary direction to that force upon the baſe, is therefore equal to the ſum of them. The force, therefore, acting upon the baſe of the wedge, is to the ſum of the forces acting upon its ſides as D F + *d f* to D E + *d e* or (by the ſimilar triangles) A G + G B that is, A B to A C + C B.

Corol. The velocities of the Wedge, and of the body refifting it, reckoned in the perpendicular direction before explained, are to each other reciprocally, as the force acting upon the bafe to the force acting upon the fides of the wedge, when thefe forces are in *equilibrio*.

Plate I. Fig. 6. For when the Wedge A B C is driven up to the top, or is in the fituation *a b c*, it is evident, that the parts of the body that is cleaved have receded from each other the length *g d* or G D, in the direction of the line perpendicular to A C or *a c*; G C therefore is the velocity of the Wedge, and G D the velocity of the refifting body. But (by the fimilar triangles) G C is to G D, as A C to A G, that is, as A C + C B to A B. And the proportion will be evidently the fame, whatever fituation the Wedge be in, between the parts of the body to be cleaved by it.

Of the SCREW.

Plate I. Fig. 7. *A Definition.* If the plain of the triangle A B C (whofe hypothenufe reprefents fuch an inclined plain as was explained above in the fecond propofition) be conceived to be fo fitted to the concave fuperficies of a hollow cylinder, (the circumference of whofe bafe is equal to the line B C) that, the plain A B C coinciding with the fuperficies of the cylinder, the line B C may be bent into the periphery of a circle equal and parallel to the circumference of the bafe; the line B A will form a kind of fpiral, afcending upon the cylindrical fuperficies, and furrounding it once. So likewife if feveral planes, fuch as A *a c*, equal and fimilar to the former, and whofe right angles are fubtended by the line B A produced, be imagined to be fitted in the fame manner to the fame fuperficies, diftant from each other by the fpace A C or *a c*, (their common height) there will be many fpirals formed by the lines A *a*, &c. all continued from one to another, and each of them once furrounding the cylindrical fuperficies. Further, if other planes fimilar and equal to A B C, be conceived in the fame manner to be fitted to the gibbous fuperficies of another cylinder, whofe bafe is equal to the bafe of the concave fuperficies of the former cylinder; there will by this means be fpirals formed in this gibbous fuperficies, exactly like thofe in the concave one before. Now if the latter cylinder, which may be turned about its axis by means of a leaver paffing

paſſing through the center of either of its baſes, and lying in the plain of that baſe, be imagined to be ſo placed within the former cylinder, which is fixed and immoveable, that, the ſuperficies agreeing, the ſpirals formed in each ſuperficies may agree with one another alſo; and if it be ſo contrived that they ſhall always thus agree, when the internal cylinder is turned about its axis, and its baſe recedes from or approaches to the baſe of the external cylinder; it is evident that two Screws, the male and the female, may be conceived to be thus generated.

Prop. 4. In the Screw, as the altitude of one ſpiral is to the circumference of the circle, whoſe radius is the leaver by which the internal cylinder is turned round; ſo is the force, perpendicularly applied to the end of that leaver, to the weight lifted up by the ſcrew, when the force and the weight are in *equilibrio.*

Dem. Let the axis of the ſcrew be perpendicular to the horizon; Plate I.
and the poſition of the leaver, by which the internal cylinder is turned Fig. 8.
about its axis, will be horizontal. Let the weight be placed any where in the line of the axis; and then that weight, by means of the internal cylinder, will preſs with equal force (in directions perpendicular to the horizon) upon every individual point of the ſpirals of the external cylinder; and the ſum of the forces with which all theſe points are preſſed, will be the ſame as the whole weight to be lifted up. But let us firſt conſider the force, or that part of the whole weight which preſſes upon any particular point. Now it is eaſy to ſee that the ſame force, in a horizontal direction, which is able to ſupport the weight which preſſes upon any one point of the ſpiral, upon the inclined plain of which that ſpiral is formed; that ſame force, with the ſame direction, is alſo ſufficient to ſupport the ſame weight upon the ſpiral; and that there is plainly no difference, whether this force be immediately applied to the point which is preſſed, or be in any other line touching the baſe of the internal cylinder. Let B C therefore be the circumference of that baſe, A C the radius, A G the leaver by which the internal cylinder is turned about its axis, F G H the circle deſcribed by the radius A G. Theſe things being ſuppoſed, from what has been ſaid, together with the definition

definition of a Screw, and the fourth Corollary of the ſecond Propoſition, it follows, that as the height of one ſpiral is to the periphery B C, ſo is the force applied to the point C, in a direction perpendicular to A C, to that part of the whole weight which that force ſupports upon any one point of the ſpiral. And (by the property of the leaver) as the circumference B C is to the circumference F H, (that is, as A C to A G) ſo is the force exerciſed in G to the force exerciſed in C; becauſe the directions of theſe forces being parallel, they have equal power in the leaver A C G, whoſe center is A. Therefore (equally by perturbation) as the height of one ſpiral to the periphery F H, ſo is the force which, exerciſed in G, ſupports that part of the whole weight by which any one point of the ſpiral is preſſed, to that part of the weight itſelf: And as the force which ſupports that one particular part of the whole weight is to that one particular part of the weight, ſo is the force which, acting in the ſame direction, ſupports all the parts of the weight, that is the whole weight, to all thoſe parts together, that is to ſupport the whole weight. Therefore, &c. *Q. E. D.*

Corol. The circular velocity of that force by which the Screw is turned round, and the velocity of the weight which is lifted up by means of the Screw, are to each other reciprocally as thoſe forces when they are in *equilibrio*. For it is evident, that in a whole revolution of the leaver, the weight is raiſed juſt the height of one ſpiral, and that in every part of the revolution the weight is raiſed proportionably.

Of the PULLEY or WINDLESS. Prop. V.

It is evident, that the Pulley may be accounted for in the ſame manner as the ballance or leaver, in which the forces are employed either on the ſame ſide of the center, or on both ſides; which, when they are in *equilibrio*, are to each other reciprocally as perpendiculars, let fall from the point which repreſents the center of the leaver, to their directions. And hence the forces of engines, which conſiſt of many Pullies, according as they are differently framed, may eaſily be explained. If the compoſition of the Pullies, or the manner of framing

framing the Windlefs be fuch, that the ropes which are fitted to the pullies are parallel to one another; and the weight be fo fufpended in the midft of the ropes, as to draw every one of them with equal force; it is felf-evident, that the force is to the weight which it fupports, as one to the number of ropes. For when that force is applied to one of the ropes only, it is directly oppofed to that part only of the whole weight which draws that rope, the pin to which the Windlefs is fixed, fupporting the other parts of the whole weight.

It is alfo evident, that in this engine the force and the weight, when they are in *equilibrio*, are to each other reciprocally as their velocities when the force raifes the weight. For it is manifeft, that thefe velocities are to each other as the decreafe of the length of all the ropes which fupport the weight, taken together, to the increafe of the length of the rope to which the force is applied, in the fame time; and that juft fo much as is loft in a given time in *all* the lengths of the ropes which fupport the weight, the very fame is gained, in the fame time, in the *one* length of that rope to which the force is applied.

On the CONGRESS *of* ELASTIC BODIES.

THE weights and velocities with which two fpherical bodies, perfectly elaftic, whofe centers are moved in the fame ftraight line, meet each other, being given; to find their velocities after they have met.

In the following computation, the motion of elaftic bodies after ftriking againft each other, is fuppofed to arife from two caufes.

I. From fimple impulfe. By the force of which alone, if the bodies had no elaftic force, each body after they had met would either wholly reft, *viz.* if they meet each other with equal motion; or they would go both on together, as if they were united into one body, with the fame velocity; and the fum of their motions, (if they moved both the fame way) or the difference of their motions, (if they moved contrary ways) would continue the fame after their meeting as before.

II. From *elastic force.* Which, in bodies perfectly elastic, is equal to the force with which they are compressed; that is, when two such bodies are struck against each other, it is equivalent to that motion which one of them would gain or lose by simple impulse only. This force acts the contrary way, and therefore the motion which is produced by it must be substracted from that motion which is in the body impelling, and added to that motion which is in the body impelled, by the force of simple impulse only, in order to find their velocities after reflection.

This being supposed, let A and B be two perfectly elastic bodies, and let A either overtake B or meet it; let their velocities be *a* and *b*. Then the motion of A will be A *a*, and the motion of B will be B *b*, and the quantity of motion in them both together, if they be moved the same or contrary ways, will be $Aa \pm Bb$, which (by the first position) will be the same after their impulse as before. Now (if they had no elastic force) their common velocity after they had met, would be $\frac{Aa \pm Bb}{A+B}$, and therefore the motion of A, $\frac{A^2a \pm ABb}{A+B}$, and that of B, $\frac{ABa \pm B^2b}{A+B}$. Now if the motion $\frac{A^2a \pm ABb}{A+B}$, which remains in A after the impulse be substracted from the motion A a, which it had at first, there will remain the motion $\frac{ABa \mp ABb}{A+B}$, which the body A has lost by simple impulse only. Now if this motion be substracted from the motion $\frac{A^2a \pm ABb}{A+B}$ which is in A, and added to the motion $\frac{ABa \pm B^2b}{A+B}$ which is in B after their meeting, from the first cause only, the remainder $\frac{A^2a \pm 2ABb - ABa}{A+B}$ will (by the second position) be the motion of A, and the sum $\frac{2ABa \pm B^2b \mp ABb}{A+B}$ will be the motion of B, from both causes together, after reflection. And by dividing separately these motions by their bodies, we shall have $\frac{Aa \pm 2Bb - Ba}{A+B}$ for the velocity of A, and $\frac{2Aa \pm Bb \mp Ab}{A+B}$ for the velocity of B after reflection. *Q. E. J.* (*See Newton's Algebra, pag.* 91. *Prob.* 12.)

N. B.

N. B. It may ſo happen, that the body A, whether it overtakes B or meets it, may loſe all its motion, or be driven back the contrary to that it moved before they met. Wherefore in this caſe the quantity $\frac{Aa + 2Bb - Ba}{A + b}$ by which the velocity after reflection is expreſſed, will either become nothing (the negative and poſitive terms deſtroying one another) or negative. So likewiſe it may happen, that when the body B meets A, it may, after their meeting, either reſt or go on to be moved the contrary way to that A was moved in, before they met; and then the quantity by which the velocity is expreſſed, will either be nothing or (as at firſt) negative. But if it be driven back the ſame way that A was moved in at firſt, the quantity by which the velocity is expreſſed, will be poſitive. For ſince the velocity that way which A was at firſt moved in, is expreſſed by the ſign $+$; it is evident, that the velocity the contrary way ought to be expreſſed by the contrary ſign $-$ throughout the whole computation.

From theſe general quantities now found, by which the velocities of the bodies A and B are expreſſed, it is eaſy to deduce the laws of motion, which are obſerved by any perfectly elaſtic bodies after reflection, in any given caſe whatſoever. For example.

1. If the velocities of two bodies meeting each other, be reciprocally as their weights, in this caſe it will be $Aa = Bb$, and therefore the quantity by which the velocity of A is expreſſed, $= \frac{-Aa - Ba}{A + B}$ $= -a$; and that of B, $= \frac{Ab + Bb}{A + B} = b$. That is, each body after their impulſe will go back with the ſame velocity with which they met each other.

2. If A ſtrikes againſt B when it is at reſt, the velocity of A will be (the quantity B, and conſequently its multiples B b, &c. vaniſhing) $= \frac{Aa - Ba}{A + B}$, and the velocity of B will be $= \frac{2Aa}{A + B}$; that is, as the ſum of their bodies is to their difference, ſo is the velocity of the body A, before reflection, to its velocity after reflection. And as the

ſum

ſum of the bodies to double the impelling body, ſo is the velocity of A, before reflection, to the velocity of B after reflection.

3. If A be equal to B, and ſtrikes againſt it when it is at reſt, the velocity of A will be = o, and the velocity of B will be = a. Which ſhews that the body A, after ſtriking, will be at reſt, and the body B will be moved with the ſame celerity after the impulſe, that A was moved with before the impulſe.

4. If A and B be equal, and meet each other with unequal velocity, the velocity of A after meeting will be = − b; and the velocity of B = a. That is, each of them will return back after meeting, having changed their velocity.

5. If A and B be equal, and A overtakes B, the velocity of A will be = b, and the velocity of B = a. That is, they will both move the ſame way they did before, having changed their velocity.

Lemma. If there be three unequal quantities, A, B, C; and A be leſs than B, and B leſs than C. I ſay, (1.) that $B + \frac{AC}{B}$ is leſs than $A + C$; (2.) that $B + \frac{AC}{B}$ is leaſt of all, when B is a mean proportional between A and C.

Dem. The firſt part is evident from Prop. 25. Book 5. of Euclid. The ſecond part may be demonſtrated thus. Let M be a mean proportional between A and C; then $M^2 = AC$. Now if M and B be equal, it is $B + \frac{AC}{B} = 2\,M$ or $2\,B$. But if there be any difference between M and B, let that difference be D; and it will be $M \pm D + \frac{M2}{M \pm D} = B + \frac{AC}{B}$. But $M \pm D + \frac{M2}{M \pm D}$ is greater than 2 M, as is evident by multiplying each of them by $M \pm D$, and comparing their products together. Therefore, &c. *Q. E. D.*

6. Let there be three elaſtic bodies, as mentioned in the Lemma, A, B, C; and let A ſtrike againſt B at reſt, and after that, let B ſtrike againſt C at reſt alſo; I ſay, that by this means, the body C will acquire greater velocity than if it had been ſtruck immediately by A alone, without the interpoſition of B; and that it then acquires the

the greateſt velocity, when B is a mean proportional between A and C. (And the ſame holds true, if the motion begins with the body C.)

For by the ſecond law explained above, the velocity of C, if it were impelled by A only, and the body B not between them, will be $\frac{2Aa}{A+C}$ or $\frac{4Aa}{2A+2C}$. And by the ſame law the velocity of C, when ſtruck by the body B with that motion which was given it by A, will be $\frac{4Aa}{A+C+B+\frac{AC}{B}}$, which two fractions, becauſe they have the ſame common numerator (4 Aa) are to one another as their denominators inverſely. Wherefore the velocity of C in the firſt caſe, is to its velocity in the ſecond, as $A+C+B+\frac{AC}{B}$ to $2A+2C$. But (by the Lemma) $B+\frac{AC}{B}$ is leſs than $A+C$, and leaſt of all when A, B, and C are in continual proportion. Therefore $A+C+B+\frac{AC}{B}$ is leſs than $2A+2C$. That is, the velocity of C in the firſt caſe is leſs than its velocity in the ſecond; and this inequality is greateſt, when A, B, and C, are in continual proportion. If the motion begins at the body C, then if *c* repreſents its celerity, and be ſubſtituted in the room of *a*, the Demonſtration will be the ſame.

7. The more bodies there are of a different magnitude, between any two bodies, ſo much greater will the velocity of the laſt be; and it will be the greateſt of all, if the bodies be in a continued proportion. This eaſily follows from the preceding articles.

8. Perfectly elaſtic bodies recede from each other after reflection, with the ſame relative velocity that they approached each other with before reflection; that is, in any given time, the diſtance between the two bodies before and after their meeting, will be the ſame at the end of that time. For the diſtance of the bodies in any given time, before they meet, may be expreſſed by $a \mp b$: *viz.* the ſame quantities by which the difference of their velocities, if they be moved the ſame way, or the ſum of their velocities, if they be moved different ways, is repreſented: Alſo the ſpaces which they deſcribe

E ſeparately

ſeparately in a given time, after reflection, may be expreſſed by the ſame quantities by which their celerities are expreſſed; wherefore, f from the quantity $\frac{2Aa \pm Bb \mp Ab}{A+B}$ which expreſſes the ſpace run thro' by the body B after meeting, the ſame way that A moved before meeting, be ſubſtracted $\frac{Aa \pm 2Bb - Ba}{A+B}$ which expreſſes the ſpace run through by the body A in the ſame time, and the ſame way; the remainder $\frac{Aa \mp Ab + Ba \mp Bb}{A+B} = a \mp b$, will give the diſtance of the two bodies at the end of the given time, after reflection.

And by the like reaſoning, other laws may be found.

On the MOTION *of* FALLING BODIES.

SIR Iſaac Newton has ſhewn, that the gravity of bodies, which are above the ſuperficics of the earth, is reciprocally as the ſquares of their diſtances from its center; but the Theorems concerning the deſcent of heavy bodies, demonſtrated by Gallilæus, Hugens, and others, are built upon this foundation, that the action of gravity is the ſame at all diſtances. The conſequences of which hypotheſis are found to be very nearly agreeable to experience, becauſe the ſpaces to which bodies can be carried above the ſuperficies, are ſo very ſmall, compared with the length of the earth's ſemi-diameter, that the difference of the diſtances from its center may be looked upon as nothing. Suppoſing therefore the action of gravity to be equable, and that there is no reſiſtance in the medium through which bodies fall; the following Theorems may be thus demonſtrated.

Prop. I. The velocities acquired by a heavy body, which was at reſt 'till it began to fall, at the concluſion of any times computed from the beginning of their fall, bear the ſame proportion to each other as theſe times.

For it is evident, that in a motion performed in the ſame ſtraight line, and accelerated by equal and ſucceſſive impulſes, the velocities acquired muſt be as the number of impulſes. If therefore we imagine the time of the deſcent to be divided into infinitely ſmall and

equal

equal moments or points of time, and that the force by which the heavy body is urged downward, adds in every one of theſe moments a new impulſe to it, always equal to the foregoing one; that is, acts upon it continually in the ſame way and manner; it is manifeſt, that the heavy body may be apprehended to have received as many Impulſes while it was falling, as there are moments of time computed from the beginning of its deſcent. The velocities acquired, therefore, are as the number of moments computed, that is, as the times taken up in falling. *Q. E. D.*

Corol. In the right angled triangle A B C, if A B, A D repreſent the times of deſcent, and if B C repreſents the velocity acquired at the end of the time A B; then D E, parallel to B C, will repreſent the velocity at the end of the time A D. Plate I. Fig. 9.

Prop. II. The ſpaces run through by a heavy body, which was at reſt before it began to fall, in any times computed from the beginning of the fall, are in a duplicate proportion, both of thoſe times and of the velocities acquired at the end of thoſe times.

For it is evident, that the ſpaces which a heavy body paſſes thro' in falling, in any times whatſoever, are to one another as the ſums of the velocities with which the heavy body is carried in every one of the moments of thoſe times. Now, the preceding Corollary being granted, every one of the lines that are parallel to D E, in the triangle A D E, do each of them repreſent every one of the velocities with which the heavy body is carried in the correſpondent moments of the time repreſented by A D. (by the preceding Corol.) Therefore the ſum of theſe lines, or the triangle A D E, will repreſent the ſum of all the velocities with which the heavy body is carried in the time A D. For the ſame reaſon, the triangle A B C will repreſent the ſum of the velocities with which the heavy body is carried in the time A B. The ſpaces, therefore, run through in the times A D, A B are to each other as the triangles A D E, A B C. But theſe triangles are to one another in a duplicate proportion, as well of A D to A B as of D E to B C; that is, as well of the times of their deſcent as of their final velocities. The ſpaces therefore run through are to one another in the ſame proportion. *Q. E. D.*

Corol.

Corol. If the times, computed from the beginning of the fall, be to one another as numbers increaſing in the rank 1, 2, 3, 4, &c. the ſpaces run thro' in theſe times will be as the ſquares of theſe numbers, *viz.* as the numbers 1, 4, 9, 16, &c. and the ſpaces run through in equal contiguous times, will be as the odd numbers 1, 3, 5, 7, &c.

Prop. III. The ſpace run through by a heavy body, which was at reſt before it began to fall, in any time whatſoever, is half the ſpace which it would run through in the ſame time, with an equable motion with the velocity acquired in the laſt moment of its fall.

Plate I. Fig. 9. Let A B repreſent the time of the deſcent; B C the velocity acquired at the end of it, and let the triangle A B C be completed into the parallelogram B F: It is manifeſt, that the ſpace paſſed through in the time A B, with the equable velocity B C, is rightly repreſented by this parallelogram. But the triangle A B C is half this parallelogram. Therefore, &c. *Q. E. D.*

N. B. The three foregoing Theorems are true alſo, if applied to heavy bodies deſcending upon any inclined planes; becauſe they are urged along thoſe planes by a force which is given and equable, and which is to the force of gravity, as the height of the plane to the length of it. *See page* 3.

Plate I. Fig. 10. Prop. IV. The velocity ultimately acquired in falling along any inclined plane A C, is equal to the velocity acquired in falling the altitude of it A B; and therefore the velocities ultimately acquired in falling along any inclined planes A C, A D, whoſe altitude is the ſame, are equal: And the times of their deſcent along the ſame planes, are as the lengths of thoſe planes.

From what has been already ſaid, it is evident, that in motions equably accelerated, the velocities generated in a given time, and conſequently the ſpaces run through, are to each other as the forces by which the velocity is generated.

Firſt then, let the perpendicular B P be let fall from B to A C; and the heavy body, in deſcending along A C, will arrive at P, in th ſame time that it would arrive at B, in falling from A; (for A B is to A P, as A C to A B; that is, as the force with which the heavy

heavy body is urged along A B, to the force with which it is urged along the plane A C;) wherefore the velocity in B is alſo to the velocity in P, as A B to A P; but the velocity in P is to the velocity in C, in a ſubduplicate ratio (by Prop. 2.) of A P to A C; that is, as A P to A B. The velocity in B therefore is to the velocity in C, in a ratio compounded of A B to A P, and A P to A B; but this is a ratio of equality. Therefore, &c. Secondly, becauſe the time of the deſcent from A to P is to the time of deſcent from A to C, in a ſubduplicate ratio (by Prop. 2.) of A P to A C alſo; that is, as A P to A B, or as A B to A C; and becauſe a heavy body, in falling from A, will arrive at B in the ſame time as at P; therefore the time of deſcent along A B is to the time of deſcent along A C, as A B to A C. And for the ſame reaſon, the time along A D, as A B to A D. Therefore, &c. *Q. E. D.*

Prop. V. If the diameter A B of any circle be erected perpendicular to the horizon, the times of deſcent along any chords, ſuch as B C drawn from the extremity of it, are equal; and the velocities acquired in the point B are to each other as thoſe chords. Plate I. Fig. 11.

For if C D be let fall perpendicular from C to A B: Firſt, the time of deſcent from A to B is to the time of deſcent from D to B, as A B to C B. (by Prop. 2.) And the time from D to B is to the time from C to B, as D B to C B. (by Prop. 4.) Therefore the time from A to B is to the time from C to B, in the ratio compounded of A B to B C and D B to B C, or as A B × B D to B C q. But theſe are equal, and conſequently the times of deſcent are equal. Wherefore, ſince the times of deſcent along any chords are all equal to the time of deſcent through the diameter, they are alſo equal to each other. Secondly, the velocity acquired in falling from D to B and from C to B, is the ſame. (by Prop. 4.) Now this latter is to the velocity acquired in falling from A to B, as C B to A B. (by Prop. 2.) Therefore, &c. *Q. E. D.*

Corol. Hence we ſee the reaſon, why the times of the vibrations of a Pendulum deſcribing very ſmall arches of a circle, are very nearly equal; for thoſe arches differ very little from their chords, either in length or poſition.

F Prop.

Plate I. Fig. 12. Prop. VI. If a heavy body defcends from any altitude, through never fo many contiguous planes of any fort, and any inclination whatfoever A B, B C, C D; it will acquire the fame velocity at the laft, as it would acquire in falling perpendicularly from the fame height.

Let A F, D G, be drawn parallel to the horizon; let C B, D C, be produced till they meet A F in the points E and F, and let the perpendicular F G be let fall.

The heavy body, in falling from A to B, will acquire the fame velocity, as if it had come to B along E B. (by Prop. 4.) Wherefore, fince the turning out of its courfe at B is fuppofed no way to hinder its motion, it will have the fame velocity in C, as if it had defcended along E C; that is as if it had defcended along C F. (by Prop. 4.) Therefore it will have the fame velocity in D, as if it had defcended along F D: And this is equal to that which it would have had, in falling perpendicularly along F G. (by Prop. 4.) Therefore, &c. *Q. E. D.*

Corol. A heavy body, defcending in any curve, will acquire the fame velocity as it would acquire in falling from the fame perpendicular height; for a curve may be looked upon as compofed of an infinite number of ftraight lines.

Plate I. F. 12, 13. Prop. VII. If the inclination of any number of contiguous planes whatfoever, A B, B C, C D, a b, b c, c d, be in the fame, and the ratio of their lengths be the fame; the times in which they will be run through by a heavy body, will be in a fubduplicate ratio of thofe lengths taken together.

Let A F, a f, be drawn parallel to the horizon; and let B C, C D, b c, c d, be produced till they meet A F, a f, in E and F, e and f. It is evident, by the hypothefis, that B E has the fame ratio to b e, and C E to c e, and D F to d f, as A B has to a b, or B C to b c, or C D to c d; and alfo as A B + B C + C D has to a b + b c + c d. Now becaufe of the equal angles B A E, b a e, the times of the defcents along A B, a b, will be in a fubduplicate ratio of A B to a b; (by Prop. 2.) and the velocities in the points B and b will be in the fame as would have been acquired in falling along E B, e b. (by Prop.

Prop. 4.) If therefore the motion be continued, the ſpaces B C, b c will be run through in the ſame times as if the heavy body had begun to fall from the points E e. But the times of the deſcents, as well through E B, e b, as through E C, e c, are in a ſubduplicate ratio of thoſe lines; that is, in a ſubduplicate ratio of A B to a b. Therefore (by diviſion) the times along B C, b c, after having fallen along A B, a b, are in the ſame ratio. And therefore (by compoſition) the times along A B + B C + C D, a b + b c + c d, are in the ſame ratio alſo. In the ſame manner it may be demonſtrated, that the times of paſſing through A B + B C + C D, a b + b c + c d, are in the ſame ratio of A B to a b, or of A B + B C + C D, to a b + b c + c d, and ſo on for ever, let the number of planes be never ſo many. Therefore, &c. *Q. E. D.*

Corol. 1. The times in which a heavy body runs through ſimilar parts of curves, whoſe poſitions are the ſame, are in a ſubduplicate ratio of thoſe parts. For thoſe parts of curves may be looked upon as compoſed of an infinite number of ſtraight lines, whoſe ratio is given, and their inclination to each other ſimilar.

Corol. 2. The times in which pendulums, deſcribing ſimilar arches of circles, vibrate, are in a ſubduplicate ratio of the lengths of the threads; for theſe threads or radius's of circles are in the ſame ratio as their ſimilar arches. And the ſame holds true, though the arches be not ſimilar, provided they be very ſmall. (by Corol. Prop. 5.)

On the MOTION *of* PROJECTILES.

THE ſame law of gravity being ſuppoſed as before, and that there is no reſiſtance from the medium, and that heavy bodies deſcend perpendicularly to a given horizontal plane; (which hypotheſis, becauſe of the ſmall ſpaces through which bodies are projected, compared with the earth's circumference, differs very inſenſibly from the truth) the affections of the motion of Projectiles may eaſily be demonſtrated.

Prop. VIII. If a body goes along with a compound motion, conſiſting of an equable motion in a ſtraight line given in poſition, and of the motion ariſing from the force of gravity; it will deſcribe a parabolick

parabolick curve, which the ſtraight line given in poſition will touch in the point where the body begins to move, and all the diameters of this curve will be perpendicular to the horizon.

Plate I. Fig. 14. Let the body be moved from the point P, with an equable motion according to the direction of the line P L given in poſition; and at the ſame time let it be drawn downwards by its own gravity, according to the direction of P G, perpendicular to the horizon P H. Now ſince neither of theſe motions hinder the other ſo, but that the body may go on according to the direction of the line P L in the ſame manner, as if the force of gravity did not act at all; and that it may likewiſe deſcend according to the direction of the line P G in the ſame manner as if it had not been impelled by the projectile motion: If the body moves through the ſpaces P L, P l with an equable motion, in the ſame times as it would fall through the ſpaces P G, p g; it is manifeſt, that if G V, g v, be drawn parallel to P L, and L V, l v, parallel to P G, till they meet each other in the points V, v, the body will be found at the end of thoſe times in the points V, v. Now becauſe the motion along the line P L is equable, P L, P l, will be to each other as the times in which they are paſſed through; but P G will be to P g as the ſquares of thoſe times. (by Prop. 2.) P G therefore, or L V, is to P g or l v, as P L q to P l q: All the points V v therefore, are in a parabolick curve, which P L touches in the point P, and all the diameters of which are parallel to P G; that is, perpendicular to the horizon. *Q. E. D.*

When I mention hereafter the Parameter ſingly, you are to underſtand that Parameter which belongs to that point in the curve deſcribed from whence the projection is made.

Plate I. Fig. 14. Prop. IX. The velocity with which the body is projected along the line P L, is equal to that which it would acquire in falling thro' a fourth part of the parameter.

A body with an equable motion paſſes through the ſpace P l, in the ſame time that it falls through the ſpace l v. Now if P l be taken equal to half the parameter, l v will be equal to half P l. Now the velocity acquired in falling through l v is ſuch, that double the ſpace l v, that is, the ſpace P l, would be run through in the time of its

its fall. (by Prop. 3.) But the body by the projectile motion paſſes through the ſame ſpace P l in the ſame time. So that the velocity of the one is equal to the velocity of the other. *Q. E. D.*

Corol. 1. If the velocity of the projectile motion be the ſame, the parameter will be the ſame, whatever the direction of the projection be.

Corol. 2. The velocity of a projected body, in any point of the curve which it deſcribes, is the ſame as it would acquire in falling through a fourth part of the parameter belonging to that point; and therefore the velocities of it, in different points, are in a ſubduplicate ratio of the parameters belonging to thoſe points. (by Prop. 2.) For the projected body may be conſidered in any point of the curve deſcribed, as if it began to be moved firſt in that point, according to the tangent of it, and afterwards deſcribed the reſt of the curve.

Corol. 3. The velocity of a projected body is leaſt, therefore, when it is in the axis of the curve; and is the ſame at equal diſtances from the axis on each ſide, and the greater the more remote it is from the axis: And the velocities of it, in different points, are to each other as the ſecants of the angles which the tangents to thoſe points, when produced, make with the horizontal line. For let the ſtraight line P L touch the curve in the point P, and meet any diameter V H produced in L, and let P O be an ordinate from the point P to the ſame diameter, which will therefore make the ſame angles with the horizontal line P H, as the tangent of the curve in the point V does. Now if P H be the radius, P L, P O will be the ſecants of the forementioned angles: And it is eaſy to ſhew, from the conic ſections, that theſe ſecants are to each other in a ſubduplicate ratio of the parameters belonging to the points P and V; that is, (by the preceding Corol.) as the velocities of the projected body in the points P, V. Plate I. Fig. 15.

Corol. 4. Let the projected body begin to move from the point A, according to any direction A T: Let the horizontal line A H be drawn, and A P erected perpendicularly to it, and equal to a fourth part of the parameter of any curve to be deſcribed with a given Plate I. Fig. 16.

 force.

force. On the diameter A P let the femicircle A T P be defcribed, cutting the direction of the projected body in T. From whence let T F be let fall perpendicular to P A. Now fince the projected body can run through a fpace double to P A with the velocity acquired in falling through P A, and in the fame time, (by Prop. 3.) and fince this velocity is equal to that with which the projected body goes out from the point A: (by Prop. 9.) if A P reprefents the time of falling from P to A, the projected body will be carried in the line of its direction A T, through a fpace double to A T, in the time reprefented by the line A T; and through a fpace four times the length of A T, in twice the time of A T. Let that fpace be A E, and from E let the perpendicular E H be let fall to the horizontal line. Further, in the time reprefented by A T the projected body will fall through the fpace F A, (by Prop. 2.) and in the time reprefented by double A T, it will fall through four times the fpace F A, or through the fpace E H: That is, in the fame time that the body by its projectile motion paffes through the fpace A E, it will fall through the fpace E H, and fo meet the horizon; but A H is its horizontal fpace, and A F the altitude of the parabola defcribed. Whence the following confectaries flow alfo.

Corol. 5. The horizontal fpaces defcribed by a projected body with a given force, are to each other as the fines of double the angles, which are made by their directions and the horizontal line; and therefore its greateft horizontal fpace is, when that angle is half a right angle, and it is equal to half the parameter of the curve defcribed; and thefe fpaces are equal, when the directions of the projected body differ from a right angle by equal angles on each fide: For thefe fpaces are as the lines F f; and if C T be the radius, F T is the fine of the angle F C T, which is double E A H, whence the reft are manifeft.

Corol. 6. The altitudes of the curves defcribed, are to each other as the verfed fines of the aforefaid angles, for they are equal to the lines F A.

Corol. 7. The times which a projected body takes up in defcribing thofe parts of the curves, which are cut off by the horizontal line, drawn

drawn through the point where the projection is, are to each other as the fines of the angles which the directions make with the horizontal line; for they are to each other as the lines A T, which, if P A be the radius, are as the fines of the angles A P T, or E A H.

Prop. X. The horizontal diftance, P H, of any point V in the curve which the projected body defcribes from the point P, where the projection is made; its perpendicular diftance from the horizon V H; and the angle L P H, which the direction of the projected body makes with the horizontal line, being given; to find the parameter and the velocity of the projectile motion. Plate I. Fig. 15.

P H and the angle L P H being given, P L and L H are given; wherefore becaufe V H is given, V L is alfo given; therefore $\frac{PLq}{LV}$ the parameter is alfo given. And fince the fpace which a body falls through in a given time, is given; viz. $16\frac{1}{2}$ London feet, in a fecond of time; it is eafy to collect from Prop. 2. what the time of the defcent through the given line L V is; that is, the time in which the given line P L is run through by the projectile motion. *Q. E. J.*

Prop. XI. Let B be a mark or any given point; let B D be its perpendicular diftance from the horizontal plane, and let G D be the horizontal diftance of another given point G in the fame plane. Let G B be joined; and from the point G, let G P be erected perpendicular to G B; and let the angle B G P be bifected, by the ftraight line G N; now if the mark B be hit by a projection made according to any direction G K; I fay, that the fame mark B will be hit by a projection made with the fame force, according to another direction G L, which makes the angle L G N with the bifecting line, equal to the angle N G K. Let the forementioned directions meet D B produced, in the points K and L. Becaufe the velocity of the projected body, according to the lines G K, G L, is fuppofed to be the fame, the times which it takes up in paffing through them, are in the fame ratio as the lines themfelves; but the fpaces which it falls through from the points K and L, in thofe times, are to each other as the fquares of the times; (by Prop. 2.) they are therefore as G K q to G L q. Now becaufe of the fimilar triangles G K B, L G B; B K is Plate I. F. 17, 18.

to,

BG, as GK to GL; and BK to BG, as BG to BL. Therefore as GKq is to GLq, ſo is BK to BL. Wherefore, ſince BK (by the hypotheſis and Prop. 8.) is the deſcent of the projected body from the point K, in the time GK, LB will be its deſcent from the point L in the time GL. Therefore (by Prop. 8.) the ſame mark B will be hit by the direction GL alſo. *Q. E. D.*

Corol. 1. If LK be biſected in F, DF will be equal to half the parameter of the curves deſcribed. For the rectangle of the parameter and LB, is equal to GLq; and the rectangle of the ſame parameter and KB, is equal to GKq. Therefore the rectangle of the ſame parameter and LK, is equal to GLq — GKq; or DLq — DKq, or to the rectangle of DL ∓ DK; that is, LK into DL ± DK. The parameter therefore is equal to DL ± DK, the half of which is DF.

Corol. 2. The nearer the directions GK, GL are to the line which biſects the angle BGP, the leſs is the force required to hit the mark B; ſo that there are no more than two directions, along which the ſame mark may be hit with the ſame force. For let the biſecting line meet BD produced in N. Now ſince the directions GK, GL are diſtant from GN by equal angles, (by Propoſition 3. Book 6. of Euclid) it is evident that the point F muſt fall higher than the point N, or DF muſt be greater than DN; and if GL and GK approach to GN, the point F ought to come to the point N; that is, the parameter will be leſſened (by the preceding Corol. 2.) and conſequently the force of the projectile motion. (by Prop. 9.)

Corol. 3. If the direction of the force with which the given mark B is hit, be the line GN itſelf, which biſects the angle BGP, then that force is the leaſt, and that direction the only one, in which the mark B can be hit with that force: And the contrary. For when GK, GL coincide with GN, the point F will coincide with N, and DN will be half the parameter. Then the reſt will follow from Prop. 11. and the preceding Corollaries.

Corol. 4. Hence we ſee the reaſon of the mechanical practice of directing a cannon, ſo as to hit the mark with the leaſt force. For having fixed a plain looking-glaſs perpendicular to the bore of the cannon;

Plate I.

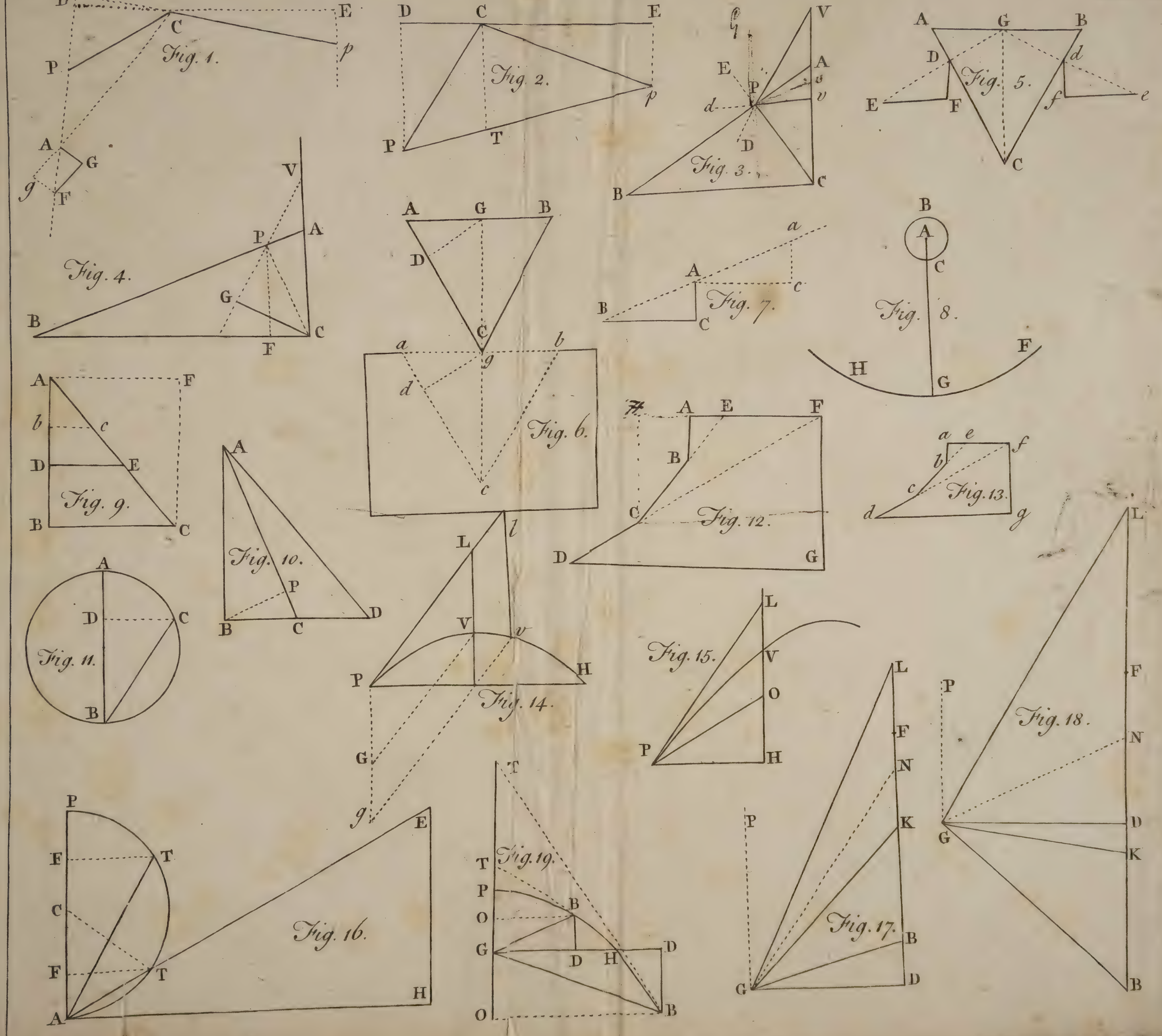
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.
Fig. 10.
Fig. 11.
Fig. 12.
Fig. 13.
Fig. 14.
Fig. 15.
Fig. 16.
Fig. 17.
Fig. 18.
Fig. 19.

cannon; let the cannon be inclined, 'till the eye, looking along a thread hanging freely with a lead at the end of it, can ſee the mark reflected by that part of the looking-glaſs over which the lead hangs. Then it is evident, from the nature of reflection and the preceding Corollary, that you have the direction required.

Corol. 5. The higheſt points which can be hit with a given force, at any horizontal diſtances, are all of them in the curve of a parabola, whoſe focus is the point from the projections are made; whoſe axis is perpendicular to the horizon, and the parameter to the axis, the ſame as that of all curves deſcribed with a given force.

For let G P H be a parabola, G the focus, G P the axis perpendicular to the horizon; the parameter to the axis the ſame as that of curves deſcribed with a given force. Let any horizontal diſtance G D be taken, and from the point D let the perpendicular D B be erected, meeting the curve in B; I ſay, the point B is the higheſt that can be hit with a given force, at the diſtance G D; or the given force is the leaſt that can hit that point. For if G B be drawn, G B$\pm$B D will be equal to half the parameter of the curve deſcribed by the leaſt force that B can be hit by. For in order to have that force hit the point B, the direction muſt biſect the angle B G P; (by Corol. 3.) then, by reaſon of that angle's being ſo biſected, and D B, G P being parallel, the triangle G B N will be Iſoſceles; and G B$\pm$ B D equal to D N, that is, to half the parameter; as is evident from that Corollary. Now in the parabola G P H, let B O be an ordinate to the axis, and let the tangent B T be drawn, meeting the axis produced in T; then (becauſe from the nature of the parabola, P O and P T, G B and G T, G O and D B are equal) G B$\pm$D B is equal to double G P, that is, (by conſtruction) equal to half a parameter of the curve deſcribed by a given force. Therefore the given force is the leaſt by which the point B in the curve of the parabola G B H can be hit: Whence the thing propoſed is manifeſt. Plate I. Fig. 19. Plate I. F. 17, 18. Plate I. Fig. 19.

Corol. 6. If D F be given, and equal to half the parameter of the curves paſſing through the point B, and from the point F be taken equal lines F L, F K, ſo as that G L, G K being drawn, they may make equal angles with the line G N, which biſects the angle B G P; Plate I. F. 17, 18.

H GL

G L, and G K will be the directions of force with which those curves passing through B will be described.

Plate II. Fig. 1, 2. Prop. XII. G D the horizontal distance of the point B, D B the altitude, and D F half the parameter, being given; to find the directions required to hit that point.

Let the perpendicular G P be erected from the point G to G D; because G D, D B are given, the angle D G B, and consequently the angle B G P is given. Let the angle B G P be bisected by the line G N, meeting D B produced in N. Now if the points F and N coincide, G N will be the direction sought. (by Cor. 3. Prop. 11.) If the point N falls above F, the point B cannot be hit at all with a given parameter, or a given force, (by the same Corol.) But if the point N falls below F, from the point F let F R be erected perpendicular to D F, meeting G N produced in R; let the line G R be bisected in S, and from the point S let S C be erected perpendicular to G R, meeting F R produced in C. On the center C, with the distance C R, let a circle be described, cutting B D produced to K and L, and if G K, G L be drawn, they will be the directions sought. For it is evident from the construction, that F L and F K are equal, and that the angles L G R, R G K are equal also; whence the rest are manifest from the 6th Corol. of the preceding Proposition. *Q. E. I.*

Plate II. Fig. 3. *The same demonstrated another way.* From the point F let F C be erected perpendicular to D F and equal to B G; and on the center C, with the distance B F, let a circle be described, cutting B D produced in the points K and L; then G K and G L will be the directions sought.

For C K q — F K q, that is, B F q — F K q (by construction) is equal to C F q or B G q. Therefore as B F — F K or B K is to B G; so is B G to B F + F K or B L. Therefore the triangles K B G, L B G are similar; (by Prop. 6. Book 6. of Euclid) therefore the angles K G B, B L G are equal; that is, if G P be erected perpendicular to G D, the angles K G B, L G P will be equal. Therefore if the angle B G P be bisected as before by the line G N, the angles L G N, N G K will be equal. Therefore (by Corol. 6. Prop. 11.) G K, G L are the directions sought. *Q. E. I.*

Corol.

Corol. 1. From the former construction there flows an arithmetical rule of solving the same problem, viz. putting S for the sine of the given angle B G P, and V for its versed sine; $V - \frac{GD}{DF} S$ will be equal to the versed sine of the difference of elevations, or of the angle L G K. The half of which angle, if it be added to and substracted from the given angle D G R, or half its supplement, to two right angles B G P, the sum and the difference will be D G L, D G K, the angles sought. Plate II. Fig. 1, 2.

For D F or G P is the sine of the arch R K G, that is, of double the angle R C S; that is (because of the common complement P R G) of double the angle P G R, or (by construction) the angle P G B. And P R is the versed sine of the same angle; and P R — P F the versed sine of the arch K R, or of the angle L G K. And it will easily appear that the angle R G D is half the supplement of B G P to two right angles. Whence the reason of the rule is evident.

Corol. 2. From the same construction flows also another arithmetical rule, by which G D, the angle B G P, and either of the elevations D G K or D G L being given, the parameter is found; for if B G P is given, R G D is given also; from whence D G K or D G L being given, R G K is given. Let v be the versed sine of double R G K, and $\frac{S}{V - v}$ G D, will be equal to half the parameter. The reason of this rule is the same as the former.

Another way. R G D and one of the elevations being given, the other is given. Wherefore as the radius is to half the sum, in one case, and half the difference in the other case of the tangents of the given elevations; so is G D to half the parameter. For D F, or half the parameter is equal to $\frac{DL \pm DK}{2}$ by Cor. 1. Prop. 11. Plate II. Fig. 1, 2.

Concerning this whole matter, see the famous Dr. Halley's Dissertation, in the Philosophical Transactions; and the learned Dr. Keil's Physics, where you will find most of these things largely demonstrated in another way.

On HEAVY BODIES *falling in a* Cycloid.

LEMMA I.

Plate II. Fig. 4. LET there be a circle defcribed on the diameter A C, which is cut at right angles by D E; from the point of the diameter A, let the ftraight line A B be drawn, meeting the circumference in B, and D E in F, and let A B be joined. I fay, A B, A D, A F are continual proportionals.

For if B D be drawn, the triangles A D B, A D F are fimilar, becaufe the angle A is common, and the angles A B D, A D F are equal, becaufe they ftand upon equal arches A D, A E. Whence the Propofition is evident.

Plate II. Fig. 5. Lem. II. Let there be any curve A H concave on one fide, and let A G be a tangent to it in the point A. Let A D be a ftraight line, any ways inclined to this tangent, and let B C, parallel to A D, cut the curve in B, and the tangent in C. I fay, if the arch A B be infinitely fmall, that arch and the part of the tangent, intercepted between the parallels A D, B C, may be looked upon as equal and coincident, and may therefore be put for each other.

Let another ftraight line touch the curve in the point B alfo, which meeting the other in E, let it be any ways produced; let F G be drawn parallel to B C, meeting each tangent produced in the points F and G; and let A B the fubtenfe of the arch be drawn.

It is manifeft, that the fubtenfe A B is always lefs than the arch, and the fum of the tangents A E, E B is greater. Now if the point B be conceived to approach to A, and during that motion the line B C is carried always parallel to itfelf; it is manifeft, that the angle B E C will be perpetually diminifhed, till it becomes lefs than any given angle whatfoever; and by that means the point F will approach nearer to G than any given diftance whatfoever, and therefore the lines E F, E G will be nearer to equality than any given difference whatfoever; that is, E F and E G may at laft be accounted as equal. Therefore E B and E C (whofe ratio to each other is the fame as E F to E G, becaufe of the fimilar triangles E B C, E F G) and alfo A E + E B and A C (A E being added to each of them)

may

may be eſteemed equal likewiſe. In the ſame manner may it be ſhewn alſo, that the ſtraight lines A B, A C, when the point B approaches to A, may at laſt be accounted equal alſo. And much more therefore may the infinitely ſmall arch A B, which is of an intermediate magnitude betwixt the ſubtenſe A B, and the ſum of the tangents A E, E B, and the tangents A C, be accounted equal.

That the infinitely ſmall arch and the tangent may be looked upon as coinciding, is evident from hence; that from the nature of curvature, there can be no ſtraight line drawn between the tangent and the curve at the point of contact.

Prop. I. Let A B C be a ſemicycloid deſcribed by the generating circle A V D; let its vertex A be turned downwards, and its axis A D be erected perpendicular to the horizon. Let any point B be taken in it, and the ſtraight line B I be drawn downwards from thence, touching the cycloid in B, and terminated by the horizontal ſtraight line A I. Let the ſtraight line F B be alſo drawn perpendicular to the axis, and on the diameter A F let the ſemicircle A F H be deſcribed. Then through any point M in the curve B A, let the ſtraight line M S be drawn parallel to B F, which will meet the circle A H F in H, and its diameter in S. Let alſo ſtraight lines be drawn, touching each curve in the points M and H. And let M N, H T, be parts of thoſe tangents intercepted between the two horizontal lines M S, N R; and let O P, a part of the tangent B I, and S R, a part of the axis D A, be included between the ſame parallels. Plate II. Fig. 6.

Theſe things being ſo; I ſay, the time in which a heavy body will run through the ſtraight line M N with an equable celerity, ſuch as is acquired in falling through the arch of the cycloid B M, is to the time that the ſtraight line O P would be run through with an equable celerity, ſuch as half that which is required in falling thro' the whole tangent B I, as the tangent H T is to the part of the axis S R.

Demonſt. From the point A to the points V and L, in which the parallels B F, M S cut the generating circle, let the ſtraight lines A V, A L be drawn, cutting the parallels M S, N R in the points

I K,

K, E, G; let A H and F H be joined; and the radius Q H of the circle A F H be drawn.

Now becauſe the ſpaces run through with an equable motion, are in the ratio compounded of the times and the velocities with which they are run through; it follows, that the times are to each other in a ratio compounded of that of the ſpaces directly and the velocities inverſely. The time therefore of running through M N, to the time of running through O P, is in a ratio compounded of the ratio of M N to O P, and of the ratio of half the celerity acquired by falling through A F, to the celerity acquired by falling through F S. (by the hypotheſis and by Prop. 4. and Corol. Prop. 6. above, concerning the deſcent of heavy bodies.) Now the whole velocity acquired in falling from F to A, is to the velocity acquired in falling from F to S, as F A to F H. (by Prop. 31. Book 3. and Prop. 8. Book 6. of Euclid; and Prop. 2. above, concerning the deſcent of heavy bodies.) Half the velocity, therefore, acquired in falling from F to A, is to the velocity acquired from F to S, as F Q to F H. The ratio, therefore, of the forementioned times, is compounded of the ratios of M N to O P, and F Q to F H. But (by the nature of the cycloid) B I is parallel to A V, and M N to A L, and therefore G L and K E are equal to M N, O P. Wherefore the forementioned ratio is compounded of the ratio of G L to K E, and F Q to F H. But G L is to E K as A L to A E; that is, as A V to A L, (by Lemma 1.) that is, as $\sqrt{AF \times AD}$ to $\sqrt{AS \times AD}$; that is, as $\sqrt{AF}$ to $\sqrt{AS}$; that is, as A F to A H; that is, as F H to H S. The ratio of the forementioned times, therefore, is compounded of the ratios of F H to H S, and F Q to F H; that is, the times are to each other as F Q or Q H to H S. But it may eaſily be made appear from Prop. 18. Book 3. and Prop. 2. and 8. Book 6. of Euclid; that Q H is to H S, as H T to S R. The times therefore of moving through M N, O P, with the forementioned celerities, are to each other as H T to S R. *Q. E. D.*

Plate II. Fig. 7. Prop. II. Suppoſe the poſition of the cycloid, the line B F, A F, B I, A I, and the ſemicircle F H A, the ſame as in the foregoing propoſition; I ſay, the time of moving through the tangent B I, with

with the equable celerity of half that which is acquired in falling through B I, is to the time of defcent through the arch of the cycloid B A, as the diameter of the circle is to half its periphery.

Demonft. Suppofe as many parallel lines as you pleafe, equidiftant from each other, to be drawn between F B and A I, which will cut the line F A in S, R, &c. the circle in H, i, &c. and the cycloid in M, r, &c. its tangent B I in O, P, &c. And from the points where each of them interfect the circle and the cycloid, let the tangents to each curve, H T, M N, i k, r s, be drawn to the following parallel, as in the figure.

The time of moving through O P equably with half the celerity acquired in falling through B I, is to the time of moving through M N, equably with the celerity acquired in falling through the arch of the cycloid B M, as S R to H T. And the time of moving through P Q, with the fame celerity as through O P, is to the time of moving through r f with the celerity acquired in falling through the arch of the cycloid B r, as R E to i k, and fo on. (by the preceding Propofition.) Therefore fince every one of the equal times of the equable motions, through the equal lines O P, P Q, &c. (by conftruction) are referred to fo many other times of motion, viz. through the tangents of the cycloid M N, r f, &c. in the fame proportion as the equal lines S R, R E are each of them referred to the tangents of the circle H T, i k, &c. the fum of the former times will be to the fum of the latter times, as the fum of the former lines to the fum of the latter lines. Let therefore the number of the parallel lines lying between F B and A I be infinite, and let the tangents to each curve be drawn in the fame manner as before; and the proportion will continue the fame. And as by this means the fum of the tangents of the circle will coincide with its femiperiphery F H A, and the fum of the tangents of the cycloid will coincide with its arch B A; and the motion through the infinitely fmall arch of the cycloid contained betwixt the two contiguous parallels, may be conceived to be the fame as that which was fuppofed through the tangents: (by Lemma 2.) It follows, that the time of defcent thro'

B I

B I with the forementioned celerity, is to the time of defcent through the arch of the cycloid B A, as the diameter F A is to its femiperiphery F H A. *Q. E. D.*

Prop. III. In a cycloid whofe axis is perpendicular to the horizon, and whofe vertex is turned downwards, the time in which a heavy body, let fall from any point of it, will arrive at the vertex, is to the time in which it would fall through the axis of the cycloid, as half the circumference of the circle is to the diameter: And therefore the times in which a heavy body, let fall from any points whatfoever, will arrive at the vertex, are equal to each other.

Plate II. Fig. 6. Let A B C be a cycloid, A the vertex turned downwards, A D the axis perpendicular to the horizon; and let a heavy body be let fall from any point B. Let B I be a tangent to the point B, meeting the horizontal line A I in I; and from the fame point B let the line B V be drawn parallel to C D, meeting the generating circle in V, and let A V be joined.

The time of defcent through the arch of the cycloid B A, is to the time of defcent in the tangent B I, with an equable celerity equal to half that which it would acquire in falling through B I, as half the periphery of the circle is to the diameter, (by the preceding Propofition.) But the time of defcent through B I is equal to the time of its defcent by a natural acceleration along the fame B I, (by Prop. 3. of the defcent of heavy bodies) or along V A, which is parallel and equal to B I, (by the nature of the cycloid.) And the time of defcent along V A is equal to the time of defcent along D A, (by Prop. 5. of the defcent of heavy bodies.) Therefore the time of defcent along the arch B A is to the time of defcent through the axis D A, as half the periphery of the circle is to the diameter.

And fince the time of falling through the axis is given, and has the fame proportion to the times of defcent through any arches of the cycloid to the vertex; it is evident that all thofe times of defcending muft be equal to each other. *Q. E. D.*

Corol. It is manifeft, that when the heavy body comes at the vertex, its motion continuing, it muft in afcending defcribe an arch of the

the cycloid in the ſame time, equal to that deſcribed in deſcending; ſo that the time of its whole motion will be to the time of its deſcent through the axis, as the circumference of a circle to the diameter.—*See Hugens's Horol. Oſcil. part* 2. *from Prop.* 16. *to the end of that part.*

The equality of the times in which a heavy body, let go from any point of a cycloid, comes to the vertex of it, may alſo be demonſtrated in the following manner.

Let a body be impelled in the line A C, towards the center C, with an accelerative force, which is every where as the diſtance from C. I ſay, that from what point ſoever of the line A C the heavy body is let fall, it will come to the center C in the ſame time. Plate II. Fig. 8.

Suppoſe any line a c unequal to A C; and let either of them, as A C, be divided into as many equal parts as you will, A B, B G, G C; let the other line a c be divided into as many equal parts, a b, b g, g c. Let us imagine the ſuppoſed force to act only in the beginnings of theſe parts, ſo that each of them may be run through with an equable motion; and let two bodies, impelled by that force, begin to be moved together, from the points A, a, towards C, c. Now becauſe the celerities with which the parts A B, a b, are run through, are as the forces with which the bodies are impelled in the points A, a; and theſe forces are to each other (by the hypotheſis) as A C to a c, or as A B to a b; therefore A B, a b, will be run through in the ſame time. Let the accelerative force act again with a ſecond impulſe in the points B, b; and becauſe the increments of the celerities are proportionable to the impulſes, or to the accelerative forces; that is, to the lines B C, b c; (by the hypotheſis) or to A C, a c; or to the celerities generated by the firſt impulſe; the whole celerities after the ſecond impulſe will be proportionable to the celerities after the firſt impulſe: Therefore the lines B G, b g, equal in proportion to the former, will be run through in the ſame time. For the ſame reaſon the lines G C, g c, will be run through in the ſame time, after the third impulſe. Let the number of equal parts in the lines A C, a c, be increaſed infinitely, and conſequently their magnitude diminiſhed in the ſame manner; ſo that the bodies may be con-

K tinually

tinually impelled by the ſuppoſed law of acceleration; and the ſame reaſoning will hold good. Wherefore in this caſe, the times of deſcent through A C, a c, are equal. Now let A B C be a cycloid, whoſe axis A D is perpendicular to the horizon, its vertex A turned downwards, and the generating circle A H D. Let the heavy body be placed in any point of it as B, and let B G be drawn perpendicular to the horizon, B F a tangent to the cycloid in the point B, and F G a perpendicular to the tangent, ſo as that they may form the triangle B G F. Let the force of gravity, whoſe direction is according to the line B G, be reſolved into two other forces B F, F G; of which two forces, it is the force B F only by which the heavy body is impelled in the point B to deſcend in the cycloid: The other force F G is taken off by the reſiſtance of the tangent or curve. Now if B H be drawn parallel to C D, and meet the generating circle in H, and A H, D H be joined; then becauſe B F is parallel to A H, (by the nature of the cycloid) and B G parallel to D A, (by conſtruction) and the angles F and H right angles, therefore the triangles B F G, A H D are ſimilar. Wherefore, as B F is to B G that is, as the force with which the heavy body is impelled in B, is to the force of gravity, ſo is H A to A D. Wherefore, becauſe the force of gravity is given, the forces with which the heavy body is impelled, in every point of the curve, are to each other as the lines A H; that is, as the arches of the cycloid A B, which (by the nature of the cycloid) are double the lines A H. The forces therefore with which a heavy body, deſcending through the arch of a cycloid, is impelled, are as its diſtances from the vertex A. Wherefore from what point ſoever it is let fall in it, it will come to the vertex in the ſame time. *Q. E. D.*

Plate II, Fig. 9.

Prop. IV. *A Problem.* To make the vibrations of a given pendulum to be all performed in the ſame time; or to make a pendulum vibrate in a cycloid.

Plate II. Fig. 10. Let C F, the given length of a pendulum, be perpendicular to the horizon; which being biſected in G, and D C I drawn perpendicular to it through C; let two ſemicycloids be deſcribed from the point C, by a generating circle whoſe diameter is C G, and let their baſes be

be C D, C I, and their vertexes A, N. Let A N be joined, which will be parallel and equal to D I, and will therefore be the baſe of a whole cycloid, deſcribed by the ſame generating circle as C B A, C N. Let this cycloid be A F N. Now if a heavy body be hanged in F, upon a thread C F or any ſuch thing which will bend, and ſo oſcillate upon the center C, between the ſemicycloids C B A, C N, that whenever it moves from the perpendicular, the upper part of the thread may bend upon that cycloid towards which the motion is made, and the remaining part, which is not applied to the cycloid, be ſtretched out in a ſtraight line; I ſay, the heavy body will always be found in the cycloid A F N.

Demonſt. Let the generating circle of the cycloid A F N be deſcribed on the axis G F; and from the point E, where the heavy body is when removed from the perpendicular, let E L be drawn parallel to A G, meeting that circle in L, and let G L be joined. From the point B, (in which the thread E B touches the cycloid C B A, the remaining part being bent upon the arch C B) let B H be drawn parallel to A G alſo, meeting the generating circle A H D in H; and let A H be joined.

The whole length of the thread C B E is equal to twice A D; (by conſtruction) therefore it is equal to the ſemicycloid C B A; (by the nature of the cycloid) and the part of the thread C B is equal to the arch C B, to which it is applied. Therefore the remaining part of it B E is equal to the remaining arch B A, and is therefore equal to twice the ſtraight line A H, (by the nature of the cycloid.) It alſo touches the cycloid in B; therefore (by the nature of the cycloid) it is likewiſe parallel to A H; therefore A H and B K are equal, and therefore B K and K E are equal alſo; therefore the parallels E L and B H are equally diſtant from A G; therefore they cut off equal arches of the generating circles, *viz.* G L equal to A H, and L F equal to H D; therefore G L and A H are parallel, and therefore G L and K E are parallel, and therefore E L is equal to K G. But K G (becauſe of the parallels H A, K B, and by the nature of the cycloid) is equal to the arch H D, that is, to the arch

L F;

L F; therefore E L is alſo equal to the arch L F; therefore (by the nature of the cycloid) the point E is in the cycloid A F N. *Q. E. D.*

Corol. 1. Since it appears that the extremity E of a pendulum vibrating between the two cycloids C A, C N, deſcribes the cycloid A F N equal to either of them; and from its ſo deſcribing it, it is manifeſt, that the very ſmall parts of the curve taken on each ſide the vertex F, do nearly coincide with very ſmall parts of the circle taken on each ſide the ſame point F. Hence it follows, that the times of the ſmalleſt vibrations of a pendulum oſcillating in a circle, are alſo very nearly equal to each other; and have very nearly the ſame ratio to the time of the perpendicular fall through half the length of the pendulum, as the circumference of a circle has to its diameter.

Corol. 2. Hence alſo appears a method of determining the ſpace through which a heavy body runs, in falling perpendicularly, in a given time. For the ratio of the time of one oſcillation, to the time of the fall through half the length of the pendulum, is given; by finding therefore the time in which a pendulum of any given length performs a ſingle vibration, the time of falling through half the length of the ſame pendulum is given. Whence (by Prop. 2. of the deſcent of heavy bodies) the ſpace which it will run through by falling in any other given time is collected.

Corol. 3. Hence alſo may be found a method of determining an univerſal and perpetual meaſure of magnitudes. For the law of gravitation, upon which the foregoing Propoſitions depend, being allowed; a pendulum of the ſame length will always, and in all places, perform ſome certain number of vibrations, in a given time. This length, therefore, may be made an univerſal and perpetual meaſure, becauſe it can always be determined by experiments. Whence it follows, that having once determined the proportion which the meaſures of the magnitudes, in any nation, bears to that length; what the quantity of thoſe meaſures is, is eaſily known at any time. Now the length of that pendulum may be determined, by obſerving how many oſcillations, in that given time, another pendulum of any length performs. For the lengths of pendulums are to each other

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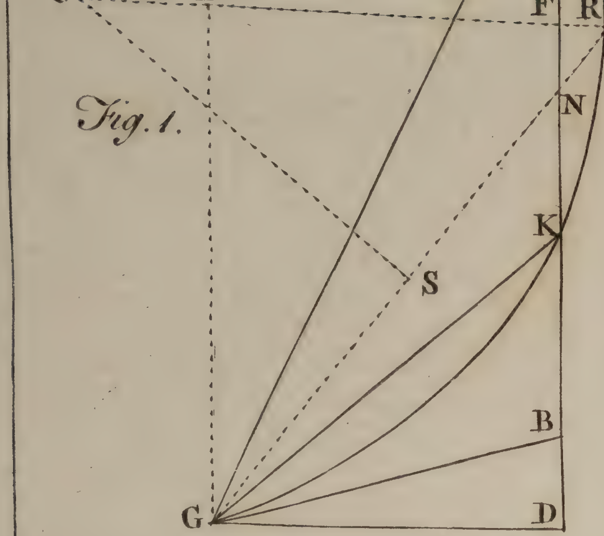
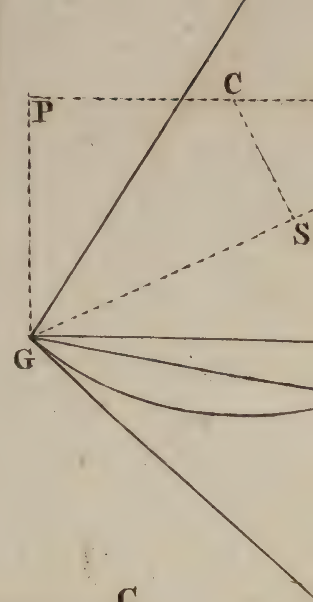


Fig. 1.



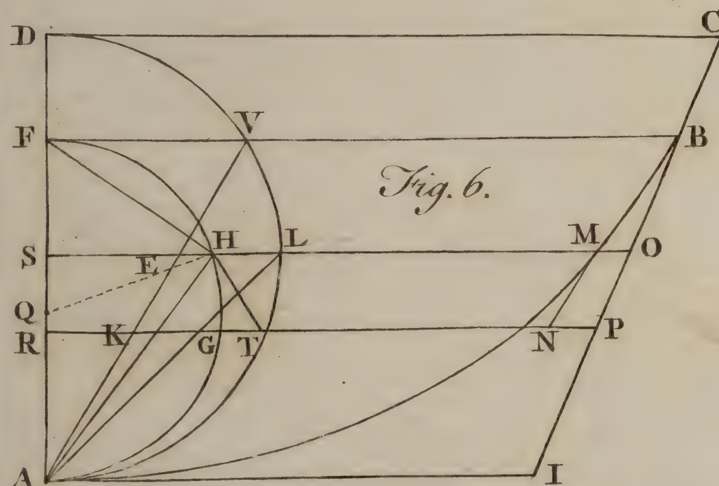
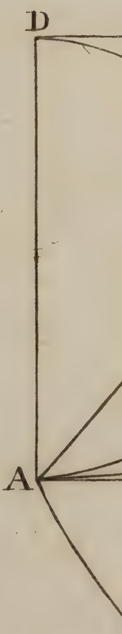


Fig. 6.



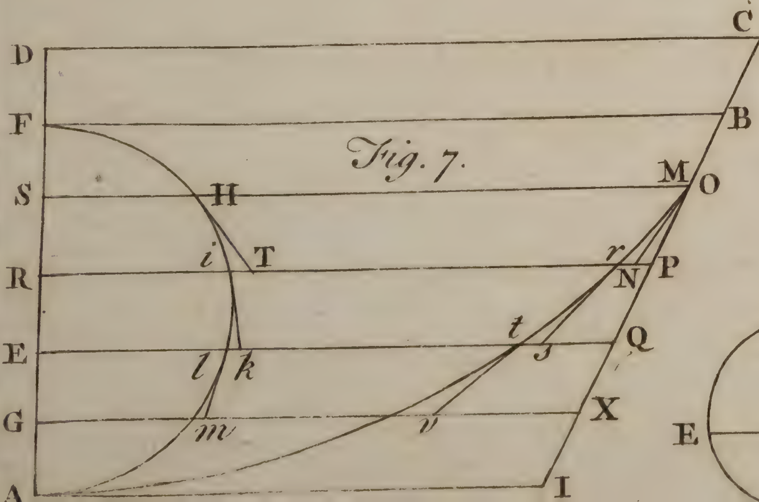
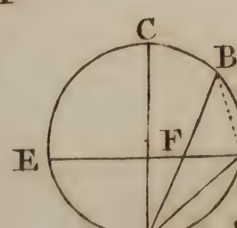


Fig. 7.



as the ſquares of the times in which a ſingle oſcillation is made; (by Prop. 3. preceding, and Prop. 2. of the deſcent of heavy bodies) and therefore they are reciprocally as the ſquare of the number of oſcillations made in the ſame time.—*See Hugenius's Horol. Oſcil. part 4. Prop.* 25. *and* 26.

On the RAIN-BOW.

IT is neceſſary, that of the rays which fall parallel and contiguous upon a refracting ſphere, thoſe that are effective or proper to produce a Rain-bow, muſt alſo come out of the ſphere parallel and contiguous; otherwiſe they will not come thick enough to the ſpectator's eye, to exhibit thoſe vivid colours of the Rain-bow. Whence it follows,

That thoſe effective rays, which come out after one reflection made by the ſuperficies of the ſphere, have all the ſame point of reflexion: Thoſe which come out after two reflections, are parallel while they are reflected; that is, from one point where they are reflected to another: Thoſe after three reflections, have all the ſame middle point of reflection: Thoſe after four, have their reflected parts, which join the ſecond and third points of reflections, parallel. And ſo on in a great many ſuch like reflections.

For let IZE be a great circle of a refracting ſphere; let the parallel and contiguous rays, and which lie in the plane of it, R I, *r i*, fall upon it; and after they are refracted, let them meet in the ſame point of the circumference Z, and then after they are reflected from thence, let them go out in the lines E M, *e m*. It is manifeſt, from the nature of the circle and of reflection, that the reflected rays Z E, *Z e*, are reſpectively equal to Z I, *Z i*, and therefore have entirely the ſame poſition with them, both with reſpect to the ſphere and to each other. Whence it follows, that ſince the refractions in E, *e*, and in I, *i*, are equal, and the incident rays R i, *r i*, parallel; the emergent rays E M, *e m*, will be parallel alſo. Whence, on the contrary, it is eaſy to ſee, that if the rays are effective, they have one and the ſame point of reflection. Plate III. Fig. 1.

Plate III. Fig. 2. For the ſame reaſon it will eaſily appear, that the effective rays RI, ri, which go out after two reflections, have their reflected parts Z Y, zy, (which connect the points of reflections Q and Y, z and y) parallel, and ought to have that poſition which was mentioned of the reflected rays in the ſeveral reflections. Whence it follows further,

That the effective rays have their angle of incidence ſo ordered, that if there be but one reflection, it's naſcent increment, or ſmalleſt increaſe, is double the increaſe of the angle of refraction made in the ſame time. If there be two reflections, the firſt increment is triple the latter. If there be three, quadruple; if four, quintuple; and ſo on.

Plate III. Fig. 1. For it is manifeſt, that the very ſmall arch I i, is the naſcent increment of the angle of incidence: And if the ſemidiameters CI, CZ be drawn; ſince CIZ or CZI is the angle of refraction, the angle iZI will be the increment of the angle of refraction generated in the ſame time, and the arch Ii double the angle iZI.

Plate III. Fig. 2 Here alſo Ii is the naſcent increment of the angle of incidence; and if the ſemidiameters CZ, Cz be drawn, ſince CZY, Czy are the angles of refraction, (becauſe ZY is parallel to zy) the angle ZCz or the arch Zz, is the increment of the angle of refraction. But 2 Zz (= arch ZY — arch zy = arch IZ — arch iz) = Ii — Zz. Therefore Ii = 3 Zz.

By much the ſame way of arguing it may be proved, that if there be three or more reflections, the ratio of the naſcent increments of the angle of incidence and refraction, is ſuch as we have aſſigned.

Wherefore, in order to find out the angle of incidence of a ray, which is effective after a given number of reflections; we muſt find out that angle, whoſe naſcent or infinitely ſmall increment bears the ſame proportion to the increment of the correſpondent angle of refraction, made at the ſame time, as the given number of reflections increaſed by unity, bears to unity. And this angle will be determined by the following Lemma.

Plate III. Fig. 3. Lemma. Let A B C be an obtuſe angled triangle, from whoſe vertex A let the perpendicular A D be let fall upon the baſe B C produced.

produced. I ſay, that the ſides A C, A B remaining the ſame, the naſcent increment of the external angle A C D, is to the increment of the angle A B C made in the ſame time, as B D to C D.

Demonſt. Imagine the ſide A C to be turned about the center A; and by this motion its extreme point C to carry the line B C D into the poſition B *c d*, ſo that the angles C A *c*, C B *c*, be the naſcent increments of the angles B A C, A B C; and let *c* C, *c D* be joined.

The angle A C D is equal to C A B and A B C; and the angle A *c d* is equal to *c* A B and A B *c*. Therefore the exceſs of A *c d* above A C D, or the naſcent increment of the angle A C D is equal to C B *c* and C A *c*. Now becauſe the angle A *c* C differs but infinitely little from a right angle, the circle deſcribed on the diameter A C, will paſs through the points D and *c*; and therefore the angles C A *c*, C D *c*, inſiſting on the ſame arch of that circle, are equal. The naſcent increment, therefore, of the angle A C D, is equal to C B *c* and C D *c*; that is, it is equal to D *c d*. But the naſcent angles D *c d*, D B *c* are to each other as their ſines; that is, as B D, the ſide of the triangle B D *c* to D *c*. Now becauſe the angle C D *c* is infinitely ſmall, D *c* is equal to D C; wherefore the naſcent increment of the angle A C D, *viz.* D *c d*, is to the increment of the angle A B C, made in the ſame time, *viz.* C B *c*, as B D to C D. *Q. E. D.*

Corol. The naſcent increments, therefore, of the angles A C D, A B D, are as the tangents of thoſe angles directly; a line being drawn from the point B, parallel to A C, 'till it meets D A produced. As appears from Prop. 4. Book 6. of Euclid.

Problem I. The ratio of refraction being given, to find the angles of incidence and refraction of an effective ray, after a given number of reflections.

Let any ſtraight line A C be taken, and let it be ſo divided in D, that A C may be to A D, as the ratio of refraction; and let it be divided again in E, ſo that A C may be to A E as the given number of reflections, increaſed by unity, is to unity. Having deſcribed the ſemicircle C B E on the diameter C E; from the center A, with the radius A D, let the arch D B be deſcribed, interſecting the ſemicircle Plate III. Fig. 4.

circle in B: Let A B, C B be drawn, then will A B C, or its complement to two right angles, be the angle of incidence, and A C B the angle of refraction required.

Demonſt. From the point A, let the perpendicular A F be let fall upon C B produced, and let B E be drawn; then will the triangles A C F, E C B, be ſimilar. Now the ſine of the angle A B C, or A B F, is to the ſine of the angle A C B, as A C to A B or A D; that is, in the given ratio of refraction, (by conſtruction.) Suppoſing therefore A B F to be the angle of incidence, A C B will be the correſponding angle of refraction. Further, the naſcent increment of the angle A B F is to the increment of the angle A C B, generated in the ſame time, as C F to B F; (by the Lemma) that is, as C A to E A; (by ſimilar triangles) that is, as the given number of reflections increaſed by unity, is to unity, (by conſtruction.) Wherefore the ratio of the naſcent increment, or the angle of incidence A B F, to the increment of the angle of refraction A C B, is ſuch as is required (by the obſervations above) in the angles of incidence and refraction of an effective ray, after a given number of reflections. The angles A B C, or A B F, and A C B, therefore, are the angles required, *Q. E. D.*

Corol. 1. From the foregoing conſtruction of this Problem, the rule of the famous Sir Iſaac Newton, for finding the angle of incidence, which you may find in his Optics, page 148, may eaſily be collected. For let I be to R in the ratio of refraction; then will $AC = \frac{I}{R} AB$; let n be the number of reflections increaſed by unity, and it will be $n\,FB = FC$. And becauſe the angle at F is a right angle, therefore $ACq - CFq = ABq - BFq$; that is, $\frac{II}{RR} ABq - nnFBq = ABq - BFq$; and therefore $nnFBq - BFq = \frac{II}{RR} ABq - ABq$; and again $\frac{BF}{AB} = \sqrt{\frac{II-RR}{nnRR-RR}}$. Whence, (if inſtead of n be put its value, which in the firſt Rain-bow is 2, in the ſecond 3, in the third 4, &c.) it will be

In

In Rain-bow the $\left\{\begin{array}{l} \text{1ſt, } \sqrt{3}\ RR: \\ \text{2d, } \sqrt{8}\ RR: \\ \text{3d, } \sqrt{15}\ RR: \end{array}\right\}$ $\sqrt{II-RR}$ :: AB : FB :: the radius : the coſine of incidence.

But the foregoing rules may be found in a more ſimple and expeditious way yet; if it be conſidered, that the ſmalleſt increments of angles or arches are to each other, as the increments of their ſines generated in the ſame time, directly, and the coſines themſelves inverſely. On the center C, with the diſtance C A, let the arch of the circle A D be deſcribed; then will D S be its ſines, and *d s* the ſine of the arch which exceeds the arch A B by D *d* the ſmalleſt difference that can be. Let D *p* be drawn perpendicular to *d s*, and *d p* will be the increment of the ſine D S generated in the ſame time. Let D C be drawn; then (by the ſimilar triangles D C S, D *d p*) it will be S C : C D :: *p d* : D *d*. Wherefore $Dd = \frac{CD \times pd}{SC}$. Conſequently (the radius C D being every where the ſame) D*d* or the ſmalleſt angle D C *d* is as $\frac{pd}{SC}$. Now the letters *n*, I, and R ſtanding for the ſame things as before, and putting Σ for the coſine of the angle of incidence of an effective ray, and σ for the coſine of the angle of the refraction of the ſame; ſince *n* is to 1 (by the obſervations above) as the ſmalleſt increment of its angle of incidence to the increment of the angle of refraction generated in the ſame time; and the increments of thoſe angles are as the increments of the ſines directly, and as the coſines themſelves inverſely; and (becauſe the ratio of the ſines of incidence and refraction is given) the increments of the ſines of incidence and refraction are to each other (by converſion) as the ſines themſelves, or as I to R: Therefore *n* will be to 1 as I to R directly, and as Σ to σ inverſely; that is, $n : 1 :: \frac{I}{\Sigma} : \frac{R}{\sigma}$. Wherefore $I\sigma = nR\Sigma$. Putting therefore *r* for the radius anſwering to Σ and σ; $\sqrt{r^2 - \Sigma^2}$ will be the ſine of the angle of incidence anſwering to that radius, and (the ratio of refraction being given) $\frac{R\sqrt{r^2 - \Sigma^2}}{I}$ will be the ſine of the angle of refraction; and there-

Plate III. Fig. 5.

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fore $\frac{\sqrt{I^2r^2 - R^2r^2 + R^2\Sigma^2}}{I}$ will be its cosine or σ. Wherefore in the equation $I\sigma = nR\Sigma$, if for σ be substituted its value, it will be $\sqrt{I^2r^2 - R^2r^2 + R^2\Sigma^2} = nR\Sigma$; and (squaring the parts and transposing them) $I^2r^2 - R^2r^2 = n^2R^2\Sigma^2 - R^2\Sigma^2$. And (resolving the equation into proportion, and extracting the roots of the terms) $\sqrt{n^2R^2 - R^2} : \sqrt{I^2 - R^2} :: r : \Sigma$ the same proportion as before. *Q. E. I.*

The foregoing rules may easily be reduced to another form, which perhaps may appear somewhat more convenient still for finding the angles of incidence and refraction, by calculation. For putting r for the radius, S for the sine of the angle of incidence, Σ for its cosine, and s for the sine of the angle of refraction. Since in the first rainbow, $3R^2 : I^2 - R^2 :: r^2 : \Sigma^2$ it will be $3R^2 : 4R^2 - I^2 :: r^2 : r^2 - \Sigma^2 = S^2$. Wherefore $S = \frac{r}{R}\sqrt{\frac{4R^2 - I^2}{3}}$. And because $S : s :: I : R$: it will be $s = \frac{r}{I}\sqrt{\frac{4R^2 - I^2}{3}}$. So likewise it will be found in the second Rain-bow that $S = \frac{r}{R}\sqrt{\frac{9R^2 - I^2}{8}}$, $s = \frac{r}{I}\sqrt{\frac{9R^2 - I^2}{8}}$. And in the third $S = \frac{r}{R}\sqrt{\frac{16R^2 - I^2}{15}}$, $s = \frac{r}{I}\sqrt{\frac{16R^2 - I^2}{15}}$. And so of the rest.

Corol. 2. The tangent of the angle of incidence of an effective ray, is to the tangent of the angle of refraction, as n to 1. It follows from what goes before, and from the Corollary of the Lemma.

Problem II. The ratio of refraction being given, and any angle of incidence whatsoever: To find the angle, which a ray of light, coming out of a refracting sphere after a given number of reflections, makes with the axis of visions or incident ray; and so to find the diameter of the Rainbow.

The angle of incidence being given, and the ratio of refraction, the angle of refraction is given. Let this angle be multiplied by twice

twice the number of reflections, increased by the number 2, and from the product let twice the angle of incidence be taken; the remaining angle is the angle fought. *Q. E. I.*

Demonst. Let C I Z E be a great circle of a sphere; in the plane of which let R I be an incident ray, which, after two refractions in the points of the circumference I and E, and one reflection between them in Z, comes out in the line E M. Let E M be produced, 'till it meets the incident ray R I, produced also, in X; and from the center C, let the semidiameters C I, C Z be drawn. Because the angles C Z I, C Z E, and also the angles Z I X, Z E X, are equal; C Z produced will pass through X, and bisect the angle I X E. The difference of the angles C Z I, Z I X, is also equal to I X Z. But C Z I or C I Z is the angle of refraction, and Z I X is the difference betwixt that angle and the angle of incidence C I X; therefore I X Z is the difference betwixt twice the angle of refraction and the angle of incidence. Consequently, the whole angle I X E is the difference betwixt four times the angle of refraction and twice the angle of incidence. *Q. E. D.* Plate III. Fig. 6.

Now let the ray R I, after two reflections in Z and E, come out in the line *e*R, meeting R I and X E (the first being refracted) in R and M; *e*EX, the external angle of the triangle *e*E M, is equal to the two angles E *e* M, *e* M E; and because the refractions in *e* and E are equal, the angles E*e*M, Z E X are equal; therefore the angles *e*E Z, *e* M E are equal: But it is evident, that the angle of reflection *e* E Z or E M *e*, is double the angle of refraction; and it has been demonstrated, that M X R is the difference betwixt four times the angle of refraction and twice the angle of incidence. Therefore the sum of the angles E M *e* or X M R and M X R, that is, the external angle of the triangle M X R, is the difference betwixt six times the angle of refraction and twice the angle of incidence. *Q. E. D.*

The same method must be proceeded in, if there be three or more reflections. But because such cases belong to the third and fourth, &c. Rain-bow, which are hardly ever seen in the heavens, because

becauſe the rays of the ſun become ſo much thinner by every reflection; and becauſe they are very eaſy, I ſhall not ſtay to demonſtrate them.

Suppoſing, therefore, that the ratio of refraction out of air into water, is what the famous Sir Iſaac Newton obſerved, (*ſee his Optics, page* 111) *viz.* as 108 to 81 in the red rays, and 109 to 81 in the blue; then, by calculation according to the foregoing rules, the diſtances of the colours from the axis of viſion (which is confirmed by obſervation) will be found to be in Rain-bow

I.	Red - - -	42.	1′.	If the ſpectator be turned from the Sun.
	Blue - - -	40.	16′.	
II.	Red - - -	50.	58′.	
	Blue - - -	54.	9′.	
III.	Red - - -	41.	37′.	If the ſpectator be turned towards the Sun.
	Blue - - -	37.	9′.	
IV.	Red - - -	43.	52′.	
	Blue - - -	49.	34′.	

Hence the breadths of the Rain-bows, and their diſtances from each other, may eaſily be collected; ſuppoſing the ſun to be only a point. But becauſe the diameter is about 30′, ſo much muſt be added to the breadth of every one of the Rain-bows; and ſo much muſt be taken from their diſtances from each other, that their true breadths and diſtances from each other may be had. 15′ muſt alſo be added to the diſtance of the outermoſt circle of colours from the axis of viſion, which paſſes through the ſun's center; and as much muſt be taken from the diſtance of the innermoſt circle, in order to have the true diſtances of thoſe circles from the axis of viſion.

Problem III. In the firſt Rain-bow, the angle which an effective ray of any kind makes with the axis of viſion, being given, to find the ratio of its refraction.

Plate III. Fig. 7. Let the angle of incidence be got: For that being found, the angle of refraction, and conſequently the ratio of refraction, will be given, (by Prob. 2. or Corol. 2. Prob. 1.) Let A B C be the angle of incidence, and, any given line C A being taken for radius, let A B be the tangent of that angle; which being biſected in D, and C D being drawn, A C D will be the angle of refraction, (by Cor. 2. Prob.

Prob. 1.) Let AE be the tangent of double this angle; and having drawn CE, the angle BCE (by Prob. 2.) will be half a given angle, and consequently will itself be given. Suppose then AE = S; BA = T; and therefore AD = $\frac{1}{2}$T; AC = r; the tangent of the given angle BCE = t; and because the line CD bisects the angle ACE, (by construction) it will be (by Prop. 3. Book 6. of Euclid) AC : CE, ($\sqrt{ACq+AEq}$) :: AD : DE. Wherefore DE = $\frac{T\sqrt{SS+rr}}{2r}$ And $\frac{T\sqrt{SS+rr}}{2r} - \frac{1}{2}T = S - T$. And again $T\sqrt{SS+rr} = 2Sr - Tr$. Then (by squaring the parts, and reduction) it will be $S = \frac{4Trr}{4rr - TT}$.

Now in order to find out T, let BF be let fall from the point B perpendicular to CE; then it will be, as the secant of the given angle BCE is to the tangent of the same; that is, as $\sqrt{rr+tt}$ to t; so is CB ($\sqrt{TT+rr}$) to $BF = t\sqrt{\frac{TT+rr}{rr+tt}}$. Again, (because the triangles EBF, ECA are similar) EC, ($\sqrt{SS+rr}$) : CA, (r) :: EB, (S—T) : $BF = \frac{Sr - Tr}{\sqrt{SS+rr}}$. Wherefore $t\sqrt{\frac{TT+rr}{rr+tt}} = \frac{Sr-Tr}{\sqrt{SS+rr}}$. Then (by squaring the parts) $\frac{TTtt+rrtt}{rr+tt} = \frac{SSrr - 2STrr + TTrr}{SS+rr}$. And (by multiplying the numerators by each other's denominators, striking out the equivalent terms, and by transposition) $SSr^4 - 2STr^4 + TTr^4 = SSTTtt + 2STrrtt + r^4tt$. And (by extracting the roots) $Srr - Trr = STt + rrt$. Now the value of S, before found, being substituted in its room, and the whole divided by $\frac{rr}{4rr-TT}$, the equation will become $T^3 = 3TTt + 4rrt$, or $T^3 - 3T^2t - 4rrt = 0$. Now by resolving this equation T will be found, and consequently, the ratio of refraction will be found from what goes before. *Q. E. I.*

Now in order to resolve this equation, let V+t be put for T, and then it will be changed into this form, $V^3 - 3Vtt - 2t^3 - 4rrt = 0$. Which being reduced by the rule, which you have briefly

demonſtrated in page 272 of the famous Sir Iſaac Newton's Algebra; and, ſuppoſing $r = 1$, and the ſecant of the given angle $\sqrt{rr+tt} - s$, it will at laſt come out $V = \sqrt[3]{t^3} + 2t + 2ts + \sqrt[3]{t^3} + 2t - 2ts$. Or $V = \sqrt[3]{t^3 + 2t \pm 2ts} + \frac{tt}{\sqrt[3]{t^3 + 2t \pm 2ts}}$. If therefore t be added to this, the ſum will be $=$ T ſought. Further, it will eaſily appear, that the ſines of the angles of incidence and refraction are $\frac{T}{\sqrt{T^2+1}}$ and $\frac{T}{\sqrt{TT+4}}$, and therefore the ratio of refraction is as $\sqrt{T^2+4}$ to $\sqrt{T^2+1}$.

But T may alſo be determined by the following conſtruction. But it is ſuppoſed that a ſtraight line of a given length may be ſo placed between two other ſtraight lines given in poſition, that when it is produced, it may paſs through a given point.—*See Newton's Algebra, page* 279, &c.

Plate III. Fig. 8. Let any ſtraight line be drawn, and in it take $CA = 4t$, and $CB = 3t$, and let BA be biſected in D; having deſcribed an arch of a circle on the center C, with the radius CD, let $DR = r$ be inſcribed in it, and let AR be joined. Having inſcribed the ſtraight line $da = DA$ between DR and AR produced, in ſuch a manner as to paſs through the point C when produced, aC will be $=$ T.

For let CG be drawn parallel to DR, and meet AR produced in G; then (becauſe the triangles GCA, RDA are ſimilar) as GC is to CA; ſo is RD to DA. And again (becauſe the triangles GCa, adR are ſimilar) as GC is to Ca, ſo is dR to da or DA. Hence CA is to dR, as Ca to DR. And (by compoſition) $Ca + CA$ is to $\left\{\begin{matrix} dR+DR, \\ dD \end{matrix}\right\}$ as CA to dR; but $dR = \frac{4rt}{T}$.

Further, $CDq - Cdq = dD \times dR$ (by Prop. 13. Book 2. of Euclid.) Whence it follows, that $\left\{\begin{matrix} Cd+CD \\ Ca+CA \end{matrix}\right\}$ is to dD, as dR is to $Cd - CD$. But $CA + Ca$, is to dD, as CA is to dR. Wherefore, as CA is to dR, ſo is dR to $\left\{\begin{matrix} Cd-CD \\ Ca-CB \end{matrix}\right\}$

Now

Now if for C A, dR, Ca, C B be ſubſtituted their values; *viz.* $4t$, $\frac{4rt}{T}$, T, $3t$; and the extreme and middle terms be multiplied by each other, and then reduced; the ſame equation will come out as before, $T^3 - 3T^2t - 4rrt = 0$. If therefore D R be radius, Ca will be the tangent of the angle of incidence. *Q. E. J.*

Corol. Hence we have a method of meaſuring the refractions of liquors, or of any other tranſparent bodies whatſoever; *viz.* by expoſing a ſphere of any ſort of tranſparent matter to the ſun, and taking by obſervation the angles which the effective rays of the firſt Rain-bow make with the axis of viſion, when they come out of it.

It may be obſerved here, that if the angle which an effective ray of a given kind, in any Rain-bow, makes with the axis of viſion, be given; the ratio of the refraction of that ray may be found, pretty much in the ſame manner as before. For the conſtruction being the ſame as then, ſuppoſe B C A to be the angle of incidence of the effective ray of any Rain-bow propoſed; and the angle E C A a multiple of the angle of refraction of the ſame ray, according to the number of reflections, increaſed by unity; then will E C B be half a given angle, or half its ſupplement, (by Prob. 2.) Whence, if C A be called r; A B, T; A E, S; the tangent of the angle E C B, t, as before; it is evident, that the ſame equation will always ariſe, $Srr - Trr = STt + rrt$; and that nothing elſe remains, but as in the foregoing Problem, to find the value of S, and to put it in its room, in that equation. Take an example hereof in the ſecond Rain-bow. Suppoſe B A to be to D A, as the number of reflections increaſed by unity, is to unity; then D C A will always be the angle of refraction, (by Corol. 2. Prob. 1.) and in the ſame Rain-bow D A = $\frac{1}{3}$ T, and the angle E C D double the angle D C A. In D A produced, let A d be taken equal to A D. Then will D C d = D C E; and then (by Prop. 3. and 22. Book 6. of Euclid) $EC^2 : Cd^2 (= CDq) :: ED^2 : Dd^2 (= 4DAq.)$ Whence $ECq - CDq : EDq - 4DAq :: CDq : 4DAq$. Alſo, $ECq = EDq + DCq + 2DE \times DA$; and $CDq = CAq + ADq$; which being ſubſtituted for ECq, C Dq, it will be $EDq + 2ED \times DA$ (= E

Plate III. Fig. 7.

$(=\overline{ED \times 2\,DA \times ED}:)\ EDq - 4\,DAq\ (=\overline{ED + 2\,DA \times ED - 2\,DA})$ $::\ ED : ED - 2\,DA :: CAq + ADq : 4\,ADq$. And therefore $ED : 2\,DA :: CAq + ADq : CAq - 3\,ADq$, or $ED : DA :: 2\,CAq + 2\,ADq : CAq - 3\,ADq$; and laſtly, $ED + DA\ (= EA) : DA :: 3\,CA^2 r - ADq : CAq - 3\,ADq$. Whence it is evident, that $EA = \frac{3\,CAq \times DA - DAc}{CAq - 3\,ADq}$. Now let S, r, and $\frac{1}{3}$ T, be put for E A, C A, and D A reſpectively, and it will be $S = \frac{r^2 T - \frac{1}{27} T^3}{r^2 - \frac{1}{3} T^2}$; and putting this value of S for S in the equation $Srr - Trr - STt - rrt = 0$, it will become $T^4 + \frac{8\,rr}{t} T^3 - 18\,rr\,T^2 - 27\,r^4 = 0$. Or (putting J for $\frac{rr}{t}$, that is, the tangent of the complement of the angle E C B) $T^4 + 8\,J\,T^3 - 18\,rr\,T^2 - 27\,r^4 = 0$.

Plate III. Fig. 9. The Problem being thus reſolved, it may be conſtructed in the following manner by means of any parabola. Let M A C be a parabola, its vertex C, the axis C D F K, the parameter of the axis R C; and taking a third part of this for the radius of a circle, let J be the tangent of the complement of the given angle E C B. Let $AD = 2\,J$ be an ordinate to the axis, and let D F be taken equal to $\frac{1}{2}$ C; $FK = 2\,CF$, and from the point K let K H be erected perpendicular to the axis, and meet the ſtraight line drawn through A and F, in H. Then having deſcribed a circle on the center H with a radius equal to $\sqrt{HAq + \frac{1}{2} CRq}$; and having let fall from the point M, where it meets with the parabola, the line M Q, perpendicular to A Q, drawn from the point A parallel to the axis; then M Q will be the tangent of the angle ſought to the radius equal to $\frac{1}{3}$ C R.

For let H K meet the ſtraight line A Q in I, and the ſtraight line M L, parallel to the axis in L; let M Q meet the axis in P alſo. Now ſince (by conſtruction) $\frac{1}{3}\,CRq = HMq - HAq$; and $HMq = \left.\begin{matrix} MLq \\ PKq \end{matrix}\right\} + LHq$; and $PKq\ (= \overline{DK \backsim DP}|^2) = DKq - 2\,DK \times DP + DPq$; and $LHq\ (= \overline{\left.\begin{matrix} LI \\ MQ \end{matrix}\right\} + IH}|^2) = MQq$

+ 2 M Q

$+ 2\,MQ \times IH + IHq$; and $HAq = \left.\begin{matrix} AIq \\ DKq \end{matrix}\right\} + IHq$.: it will be $\frac{1}{3}CRq = DPq - 2\,DP \times DK + MQq + 2\,MQ \times IH$.

Further, (from the nature of the parabola) as $ADq : MPq - ADq \left(= MQq + 2\,MQ \times \frac{PQ}{AD}\right) :: CD \left(= \frac{ADq}{CR}\right) : DP$; whence $DP = \frac{MQq + 2\,MQ \times AD}{CR}$. Alfo $DK\,(= 2\,CD + \frac{3}{2}CR) = \frac{2\,ADq}{CR} + \frac{3}{2}CR$. And (becaufe the triangles FDA, AIH are fimilar) $IH = \frac{4\,AD^c}{CRq} + 3\,AD$. Let thefe values be fubftituted in the foregoing equation for DP, DK, IH, and it will produce $\frac{1}{3}CRq = \frac{MQqq + 4\,AD \times MQ^c}{CRq} - 2\,MQq$. Or $MQqq + 4\,AD \times MQ - 2\,CRq \times MQq - \frac{1}{3}CRqq = 0$. And laftly, putting $MQ = T$, $AD = 2\,J$, $CR = 3\,r$, it will be $T^4 + 8\,JT^3 - 18\,r^2T^2 - 27\,r^4 = 0$. Whence it is evident, that MQ is the tangent of the angle fought to the radius $\frac{1}{3}CR$.

If the roots of this equation be defired in numbers, let the numeral tangent of the complement of the angle ECB in the tables, be fubftituted for J, and the numeral radius in the tables for r; and then a numeral equation will be given, which may be refolved by the common rules.

For inftance, the angle which the blue rays make with the axis of vifion in the fecond Rain-bow, is 54°. 9′. 26″. Half of this, *viz.* 27°. 4′. 48″. is the complement of the angle ECB. And the tangent belonging to it, $\left(\frac{rr}{t} = J\right)$. 5112854, fuppofing the radius (r) *s*. Thefe then being fubftituted, in the foregoing equation, for J and r; there will arife the numeral equation $T^4 + 4.0902832\,T^3 - 18\,T^2 - 27 = 0$. By refolving of which, T or the tangent of the angle of incidence, will be found to be 2.9775981; and the third part of this 0.9925327 is the tangent of the angle of refraction; and the correfpondent fines of thefe will give the ratio of refraction

of the blue rays. Now these sines are to each other, and consequently the ratio of refraction, as $\sqrt{T^2+9}$ to $\sqrt{T^2+1}$, that is, as 42268 to 31410, or as 109 to 81 very nearly.

The aforesaid equation has also a negative root, *viz.*— 6.81622765; from whence it may be gathered, that the ratio of refraction is very nearly as 347 to 321. For there are two cases of refraction, in which the effective blue rays of the second Rain-bow make the same angle (54°, 9′ ½) with the axis of vision; or when the ratio of refraction is as 109 to 81; as in rain water, in which case the tangent of the angle of incidence will be 2.9775981; or as 347 to 321. And then the tangent of the angle of incidence will be 6.8162765. And as to this latter case, if the excess of the sines of incidence of different sorts of rays, above the common sine of refraction, be supposed to be always in a given ratio; since the ratio of refraction of the blue rays is as 347 to 321, that of the red rays in the same medium will be nearly as 346 to 321. Whence it will appear by calculation according to the foregoing rules, that in such a medium the red colour will be outermost, and make an angle of about 56 ½ gr; with the axis of vision, and the blue within, in the same order as the colours of the first Rain-bow.

FINIS.

Plate III.